On hamiltonian cycles in Cayley graphs with commutator subgroup of order pq

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Example

Cayley graph

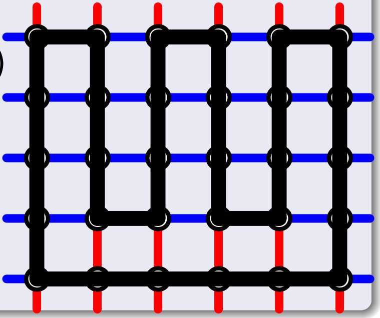
$\mathrm{Cay}(\mathbb{Z}_m \oplus \mathbb{Z}_n; \{\pm(1,0), \pm(0,1)\})$

vertices: elements of $\mathbb{Z}_m \oplus \mathbb{Z}_n$

edges: $x - x \pm (1,0)$

and $x - x \pm (0,1)$

has hamiltonian cycle



Defn. **Cay(G;S)** for group G and $S \subseteq G$ with $S = S^{-1}$

vertices = elt's of G edge $x - xs$ for $x \in G$, $s \in S$

Exercise

G abelian $\Rightarrow \forall S$, $\mathrm{Cay}(G; S)$ has a hamiltonian cycle

(if connected, i.e., if $\langle S \rangle = G$).

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Open Problem (~1970)

¿Every connected Cayley graph has a hamiltonian cycle?

Many papers on this topic
[**Marušič**, Kutnar, Šparl, Alspach, Morris^2, Gallian, . . .]

¿ Can we find a ham cycle if G is almost abelian ?

Question: What is the next best thing to abelian?
Group theorist's answer: **nilpotent**. [Moravec minicourse]

Remark. Open for nilpotent groups (but not p-groups).
(**Cubic** Cayley graphs on **nilpotent** groups have a ham **path**.)

¿ Can we find a ham cycle if G is almost abelian ?

Recall. *commutator subgroup* $G' = \langle\, ghg^{-1}h^{-1} \mid g,h \in G \,\rangle$.

G abelian $\Leftrightarrow G' = \{e\} \Leftrightarrow |G'| = 1$.

¿ Can we find a ham cycle if $|G'|$ is small ?

Theorem (Marušič, Durnberger, Keating-Witte 1985)

Cay$(G;S)$ *has a ham cycle if* $|G'| = p$ *(prime)*.

Open problem. Find ham cycle if $G' = \mathbb{Z}_2 \times \mathbb{Z}_2$.

Open Problem (Marušič 1985)

Show Cay$(G;S)$ *has ham cycle if* $|G'| = p_1 p_2$. $(p_1 \neq p_2)$

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Show Cay$(G;S)$ *has ham cycle if* $|G'| = p_1 p_2$. ($p_1 \neq p_2$)

Work in progress:

✓ G nilpotent [Ghaderpour-Morris]

✓ $|G|$ odd [Morris]

✓? $p_1 = 2$ (in progress)

Hardest case: $|G'|$ odd, but $|G/G'|$ even (and small).

Proofs use **voltage graphs**. [Ellingham minicourse]

- G/G' is abelian, so Cay$(G/G'; \overline{S})$ has ham cyc.
- Lift this to a hamiltonian cycle in Cay$(G;S)$.

$\mathrm{Cay}(G;S)$ *has a ham cycle if* $|G'| = p$

Idea of proof. Ham cyc in $\mathrm{Cay}(G/G';\overline{S})$:

$$\overline{x_0} \xrightarrow{\overline{s_1}} \overline{x_1} \xrightarrow{\overline{s_2}} \overline{x_2} \xrightarrow{\overline{s_3}} \overline{x_3} \xrightarrow{\overline{s_4}} \cdots \xrightarrow{\overline{s_n}} \overline{x_n} \quad (= \overline{x_0} = \overline{e}).$$

Then $\overline{x_i} = \overline{x_{i-1} s_i}$, so $\overline{x_n} = \overline{s_1 s_2 \cdots s_n}$.

Let $\pi = s_1 s_2 \cdots s_n \in G'$. ("voltage")

There are *many* ham cycs in G/G'.

Find one with $\pi \neq e$ ("Marušič's Method")

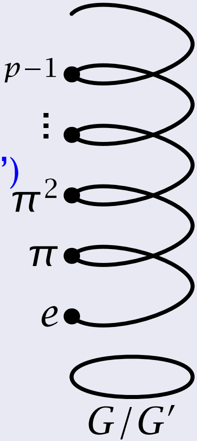
so $\langle \pi \rangle = G'$.

Ham cycle in G' lifts to a path in G

from e to π.

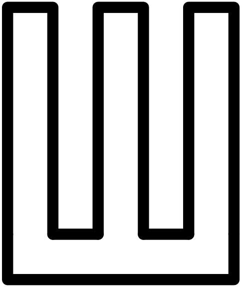
Repeated lifts extend this

to a hamiltonian cycle in G.

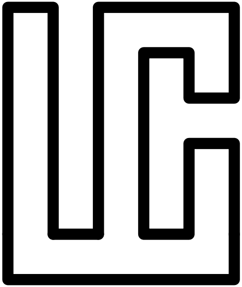


“Marušič’s Method”

Ham cycle in Cay$(G/G';\overline{S})$:



Other ham cycles:



Voltage not all same (usually), so some voltage $\neq e$.

- K. Kutnar, D. Marušič, D & J. Morris, and P. Šparl: Hamiltonian cycles in Cayley graphs whose order has few prime factors, *Ars Math. Contemp.* 5 (2012) 27–71. MR 2853700, http://amc.imfm.si/index.php/amc/article/view/177

- K. Keating and D. Witte: On Hamilton cycles in Cayley graphs in groups with cyclic commutator subgroup, in *Cycles in Graphs (Burnaby, B.C., 1982)*. North-Holland, Amsterdam, 1985, pp. 89–102. MR 0821508

- D. W. Morris: Odd-order Cayley graphs with commutator subgroup of order pq are hamiltonian, *Ars Math. Contemp.* 8 (2015), no. 1, 1–28. http://amc.imfm.si/index.php/amc/article/view/330