Realisations of {4,4} toroidal maps

Isabel Hubard

Javier Bracho
Daniel Pellicer

Instituto de Matemáticas UNAM

UNIVERZA NA PRIMORSKEM
UNIVERSITÀ DEL LITORALE

REPUBLIKA SLOVENIJA
MINISTRSTVO ZA IZOBRAŽEVANJE, ZNANOST IN ŠPORT

Naložba v vašo prihodnost
OPERACIJO DELNO FINANCIRA EVROPSKA UNIJA
Evropski socialni sklad

Motivation...

Highly symmetric polyhedra in Euclidean Spaces...

1978 Grünbaum

There are 18 finite regular polyhedra in $\mathbb{R}^3$

1982 Dress

There are 48 regular polyhedra in $\mathbb{R}^3$

~2004 Schulte

There are no finite chiral polyhedra in $\mathbb{R}^3$

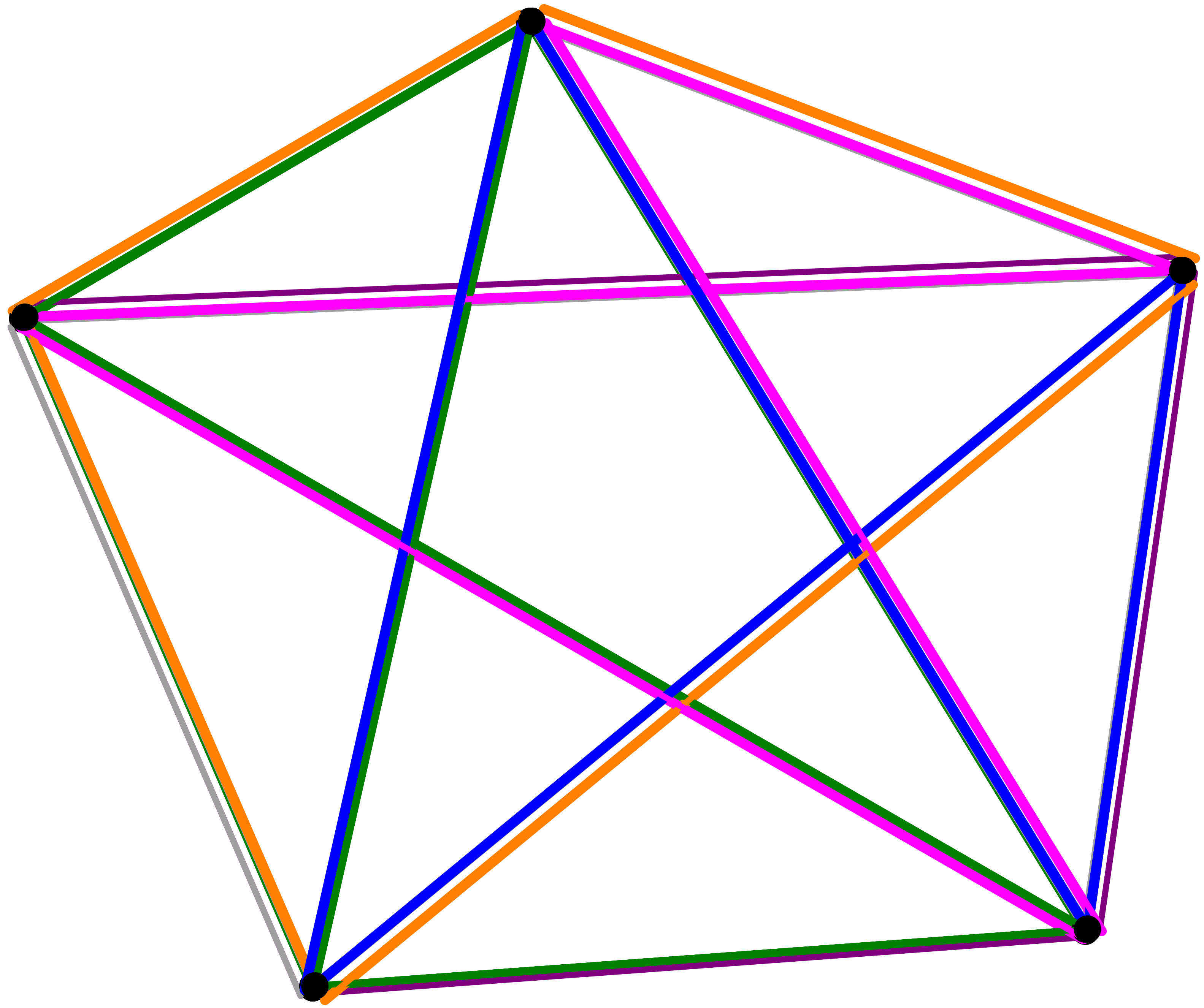
Classified chiral polyhedra in $\mathbb{R}^3$

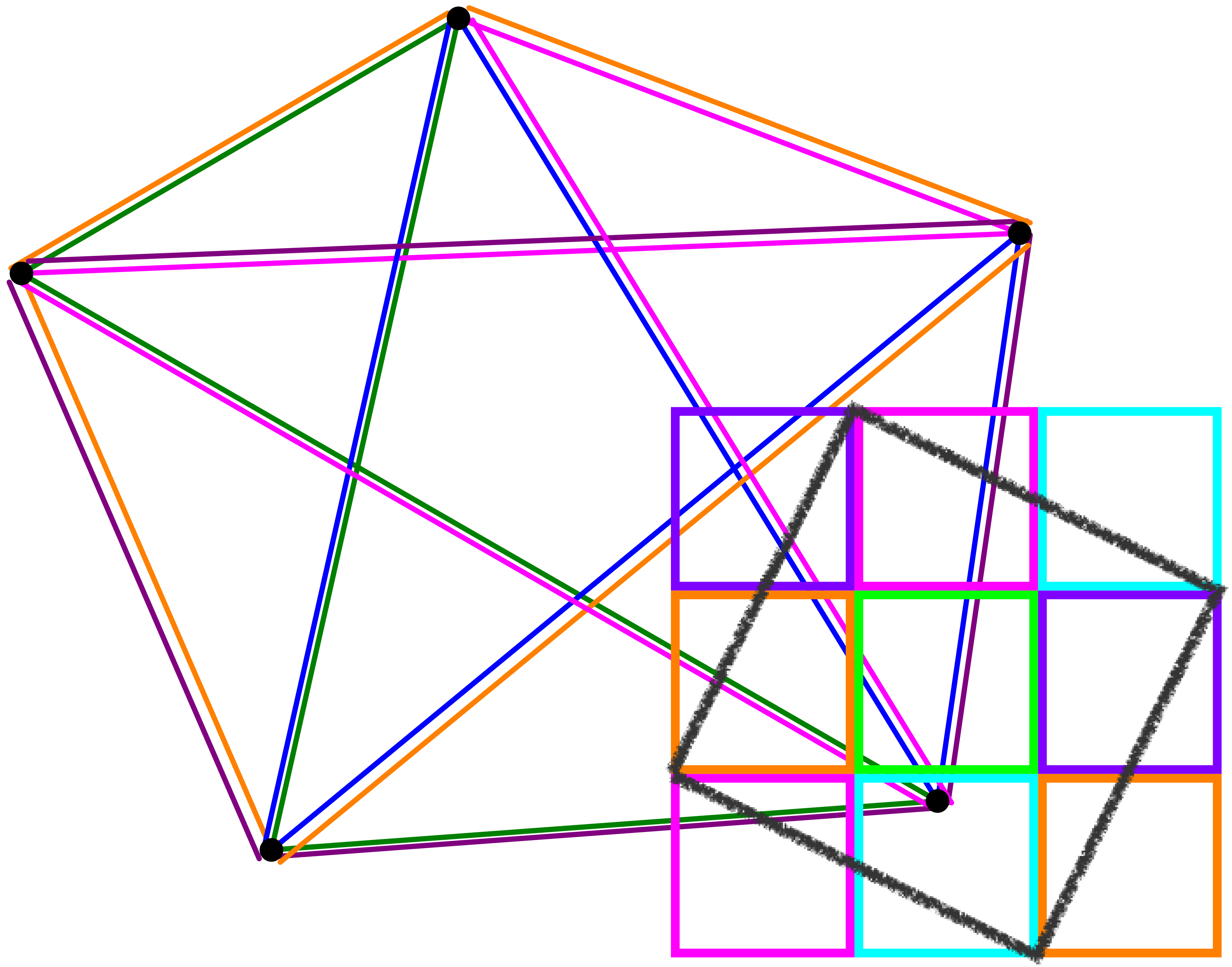
Motivation...

Highly symmetric polyhedra in Euclidean Spaces...

2007 McMullen
Classified finite regular polyhedra in $\mathbb{R}^4$

What about finite chiral polyhedra in $\mathbb{R}^4$?
What about {4,4} toroids in $\mathbb{R}^4$?





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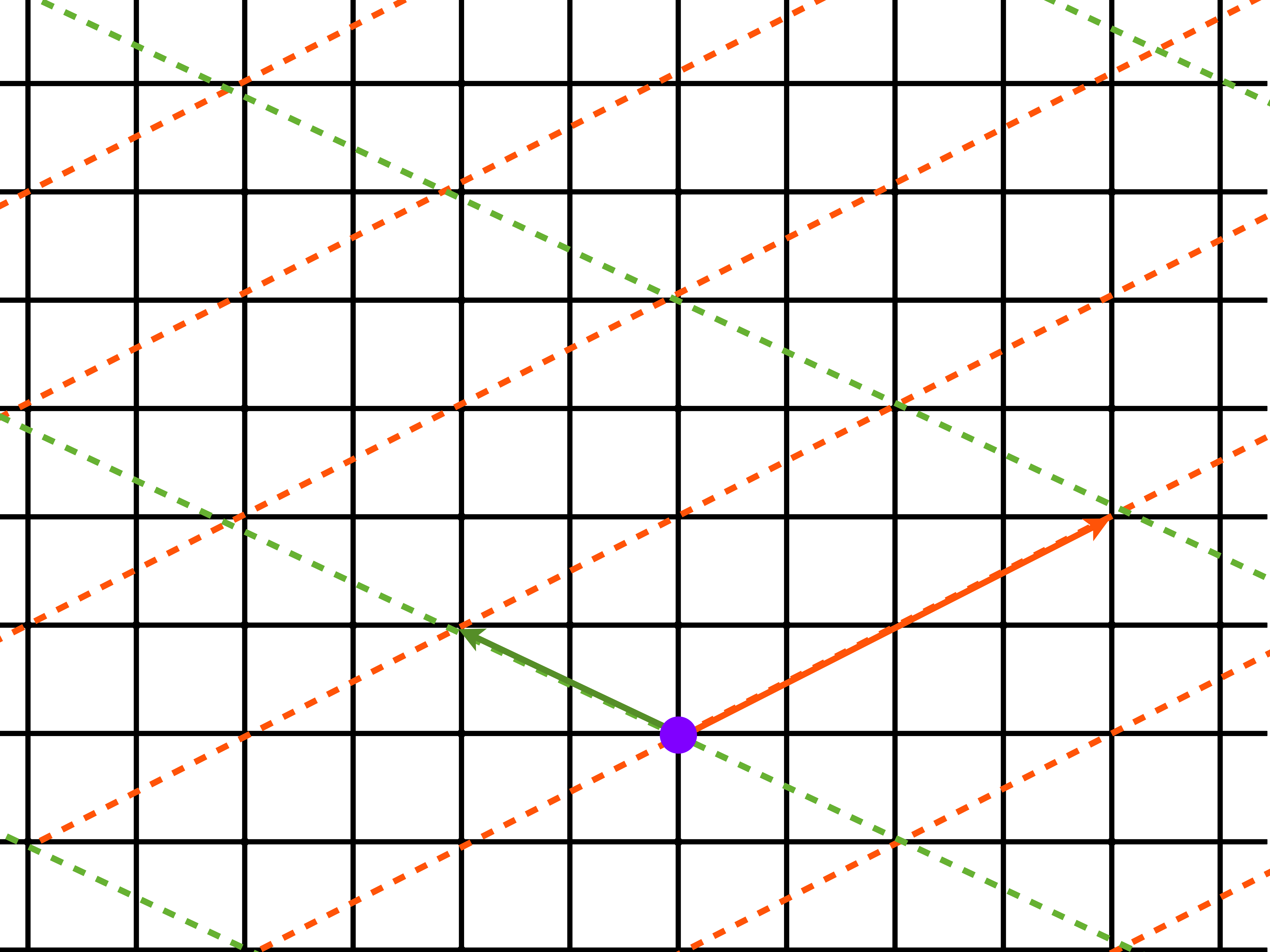
{4,4} toroids



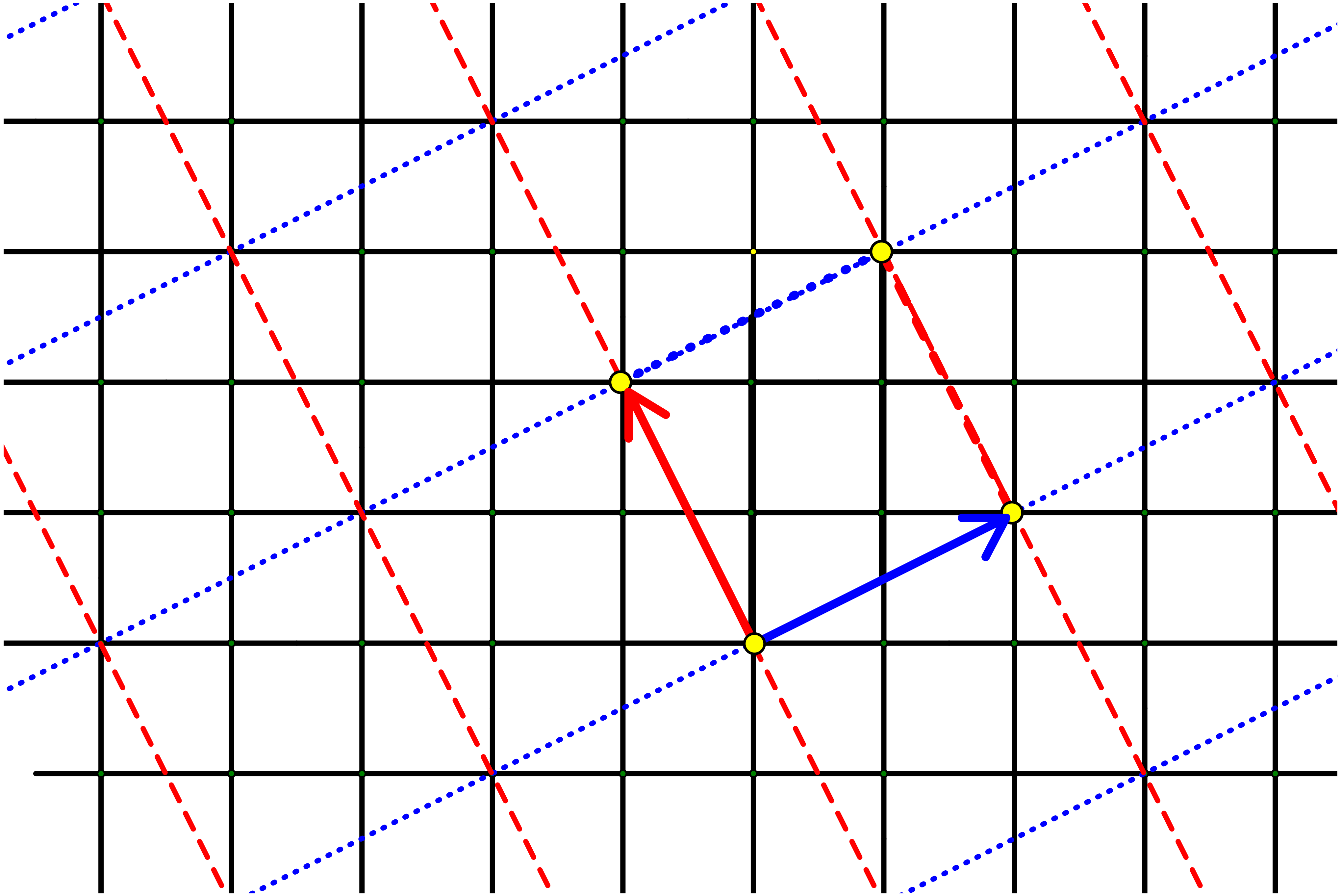
{4,4} toroids

Denote by $\mathscr{U}$ the (regular) square tessellation of the plane.

Let $\mathscr{G}$ be a translation subgroup of Aut($\mathscr{U}$).

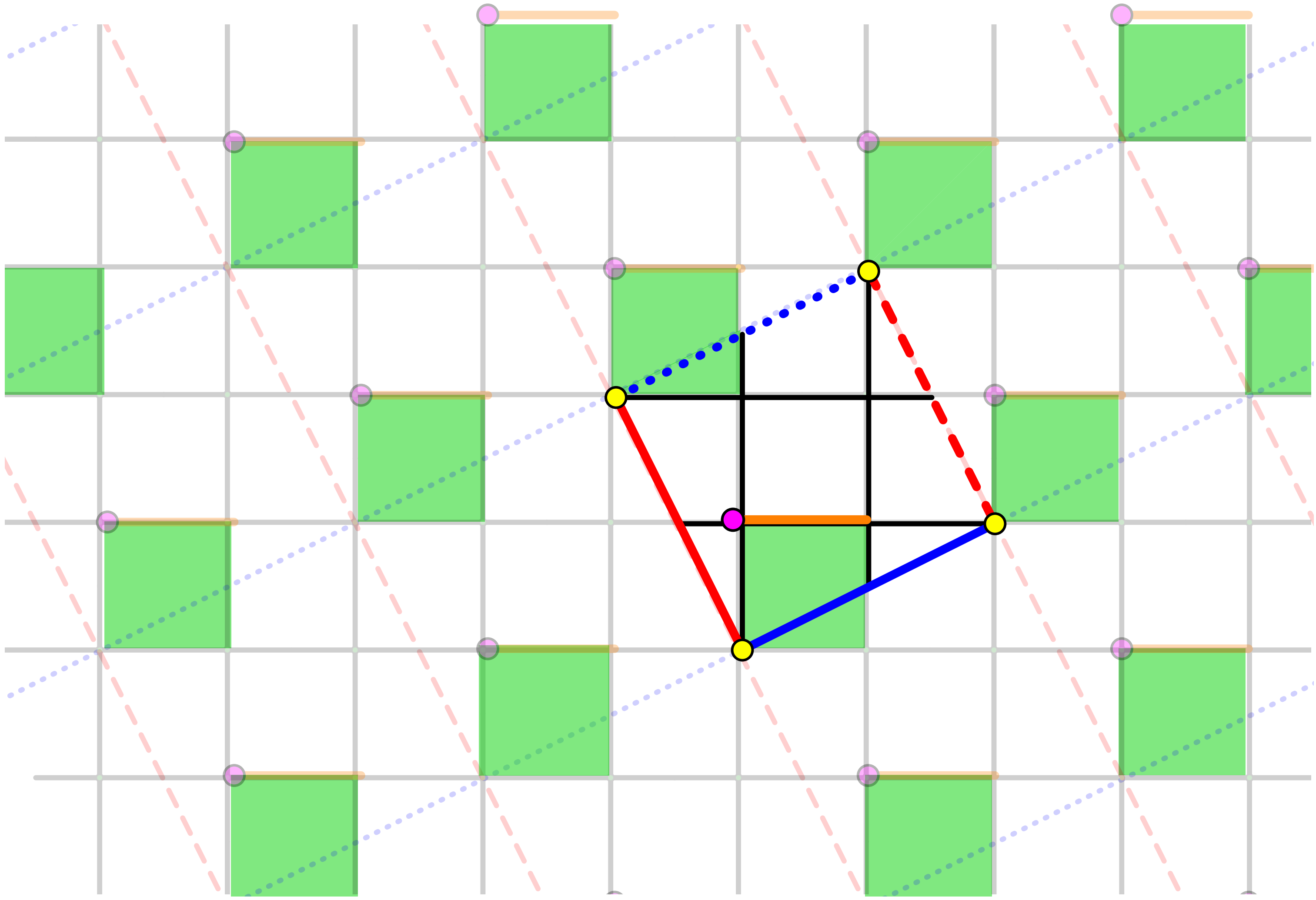
The quotient $\mathscr{U}/\mathscr{G}$ is a toroid.

{4,4} toroids

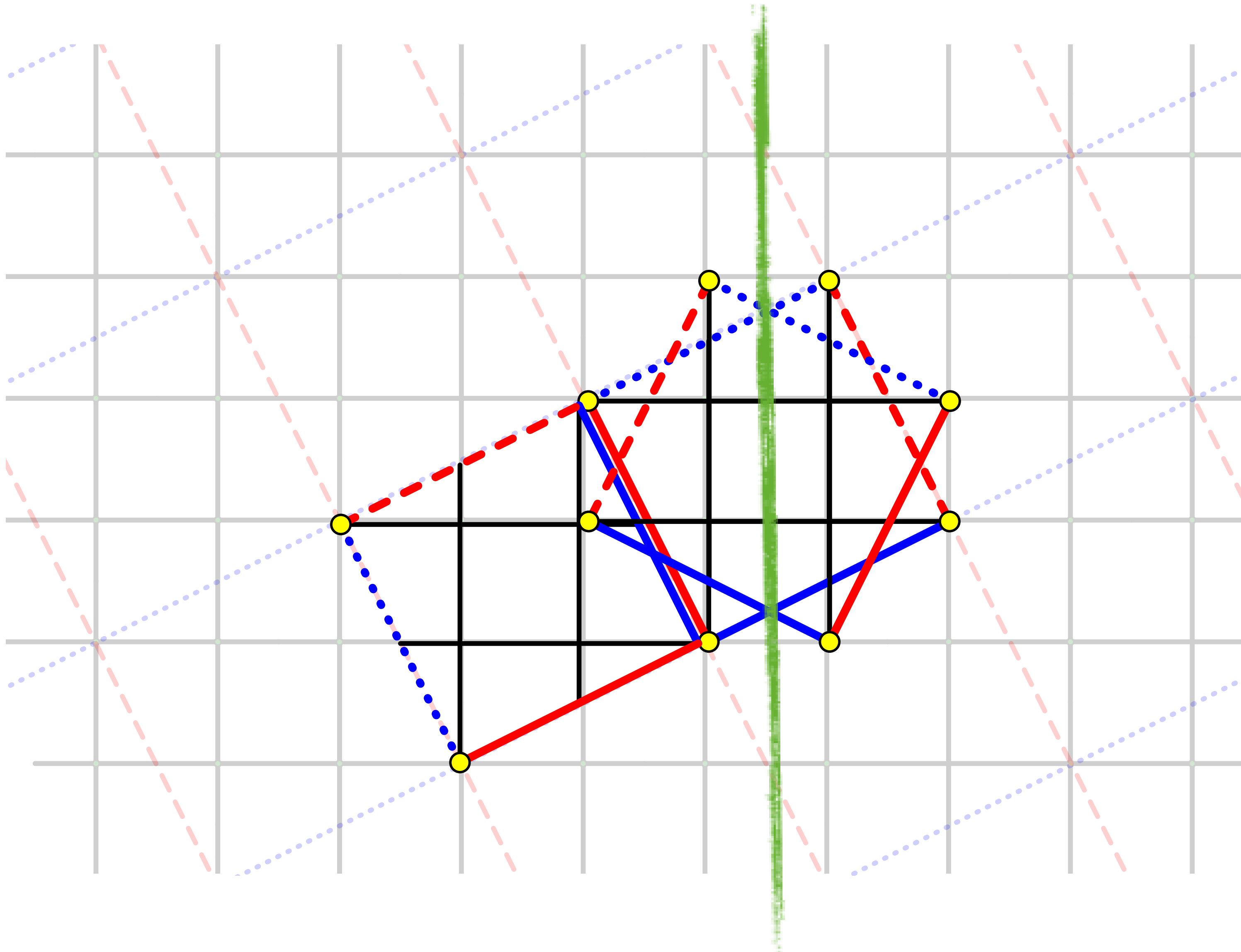


The quotient $\mathscr{U}/\mathscr{G}$ is a toroid.

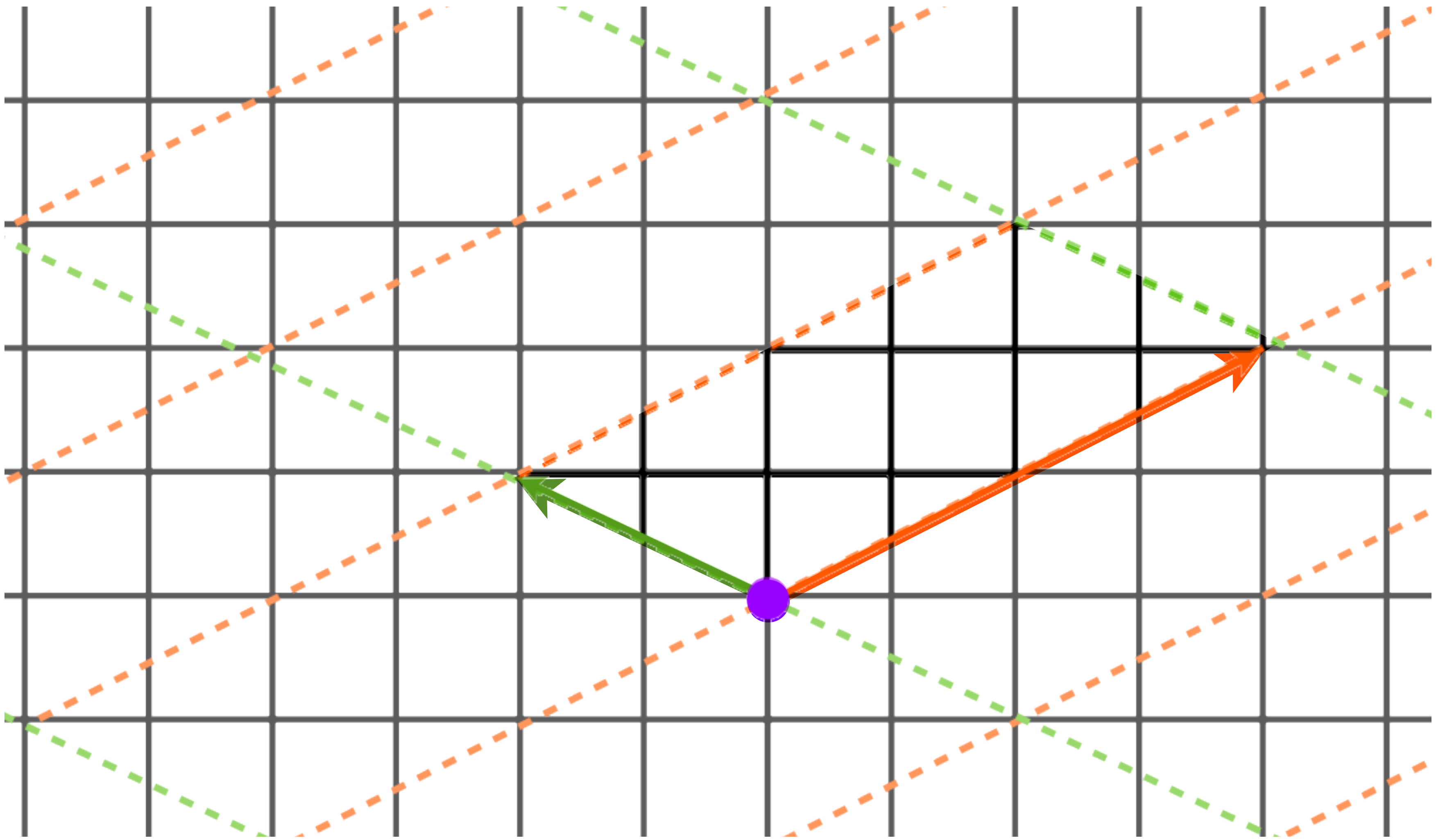
A face, edge or vertex of a toroid $\mathscr{U}/\mathscr{G}$ is the orbit F$\mathscr{G}$, where F is a face, edge or vertex of $\mathscr{U}$, resp.



A flag of $\mathscr{U}/\mathscr{G}$ is the orbit $\Phi\mathscr{G}$, where Φ is a flag of $\mathscr{U}$.



A symmetry of a toroid is a symmetry of $\mathscr{U}$ that normalises $\mathscr{G}$.



Translation (to vertices) are symmetries
Half-turns (at vertices) are symmetries

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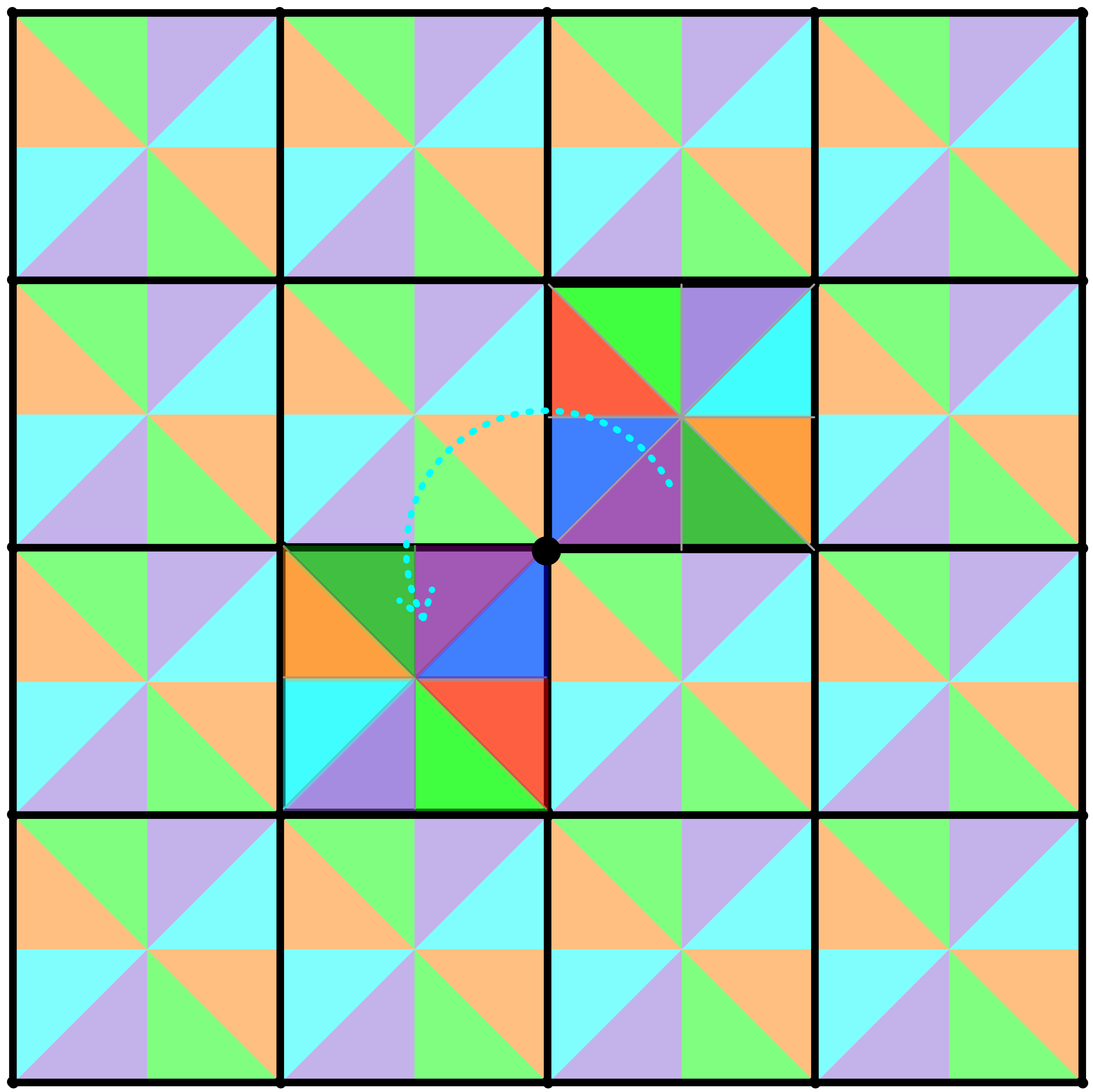
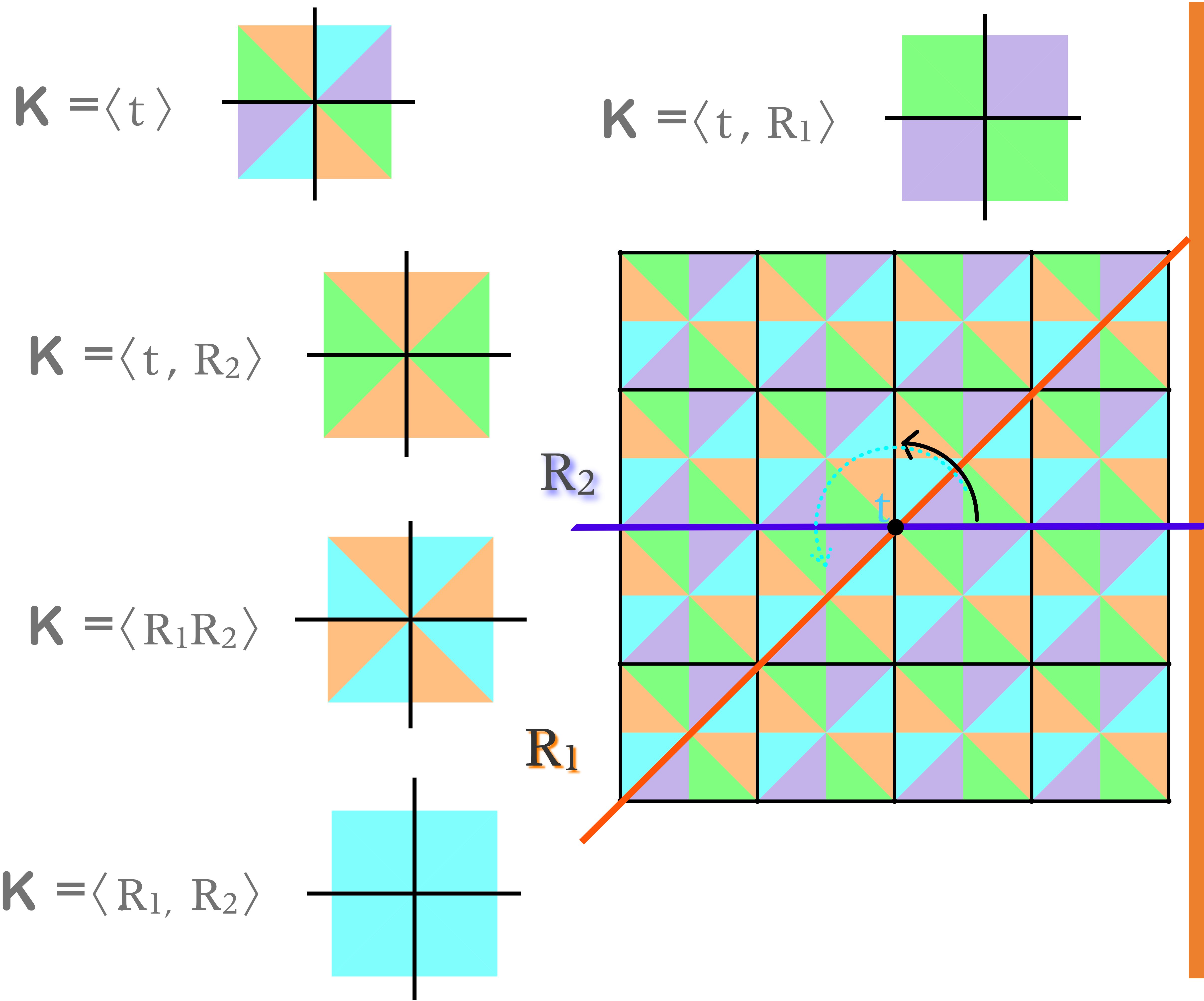
Half-turns (at vertices) are symmetries

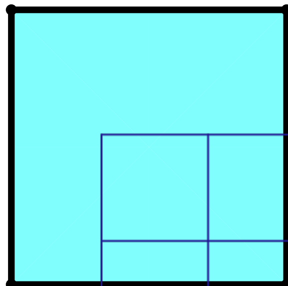




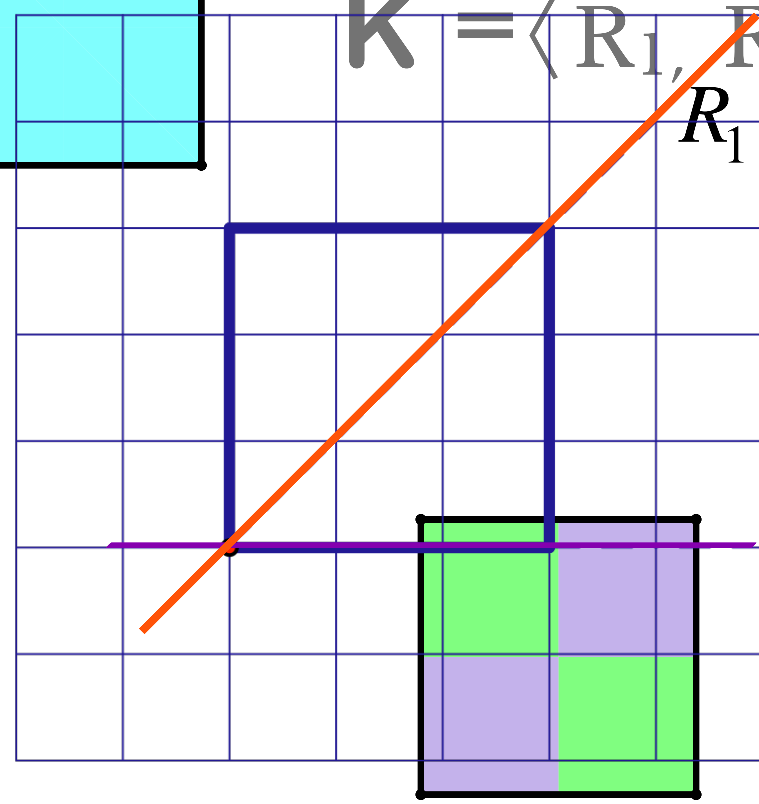
K =⟨t⟩
K =⟨t, R1⟩
K =⟨t, R2⟩
K =⟨R1R2⟩
K =⟨R1, R2⟩
R2
t
R1

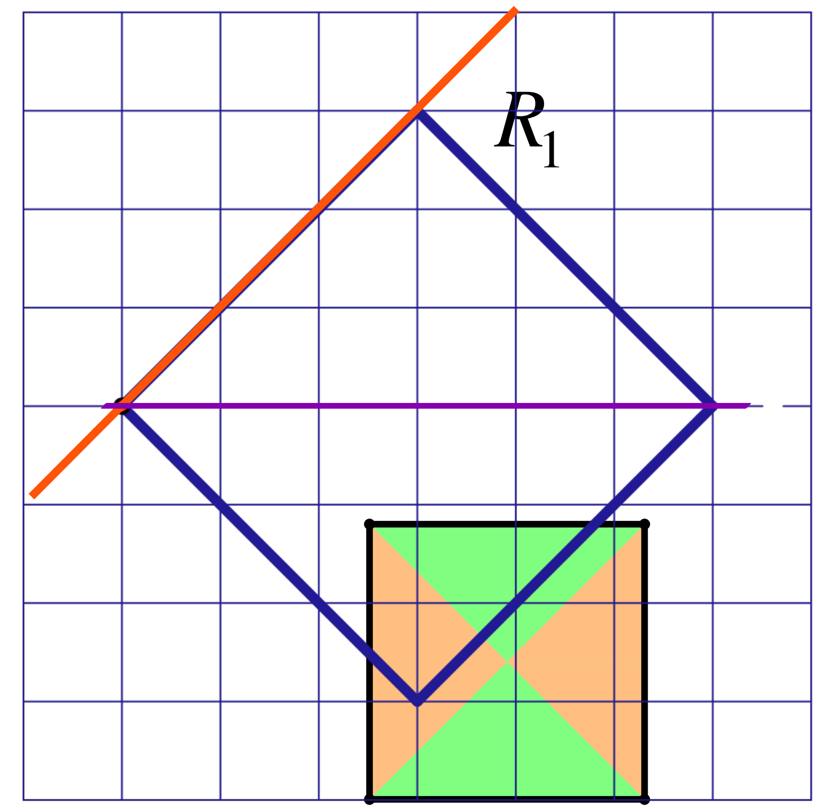
Regular {4,4} toroids

Coxeter 1948

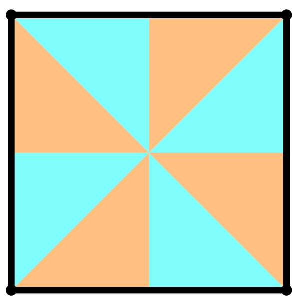


$\mathbf{K} = \langle R_1, R_2 \rangle$





R_2

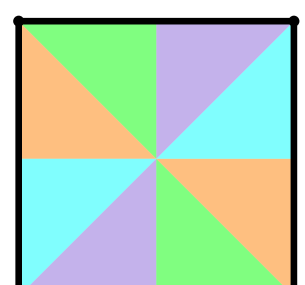


$\mathbf{K} = \langle R_1 R_2 \rangle$

$\{4,4\}_{(a,0)(0,a)}, \; a > 0$ $\mathbf{K} = \langle t, R_1 \rangle$

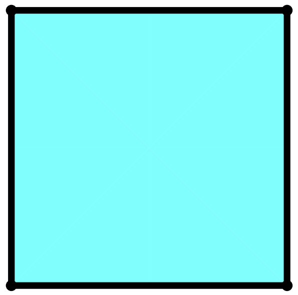
$\{4,4\}_{(a,a)(a,-a)}, \; a > 0$ $\mathbf{K} = \langle t, R_2 \rangle$

$\mathbf{K} = \langle t \rangle$



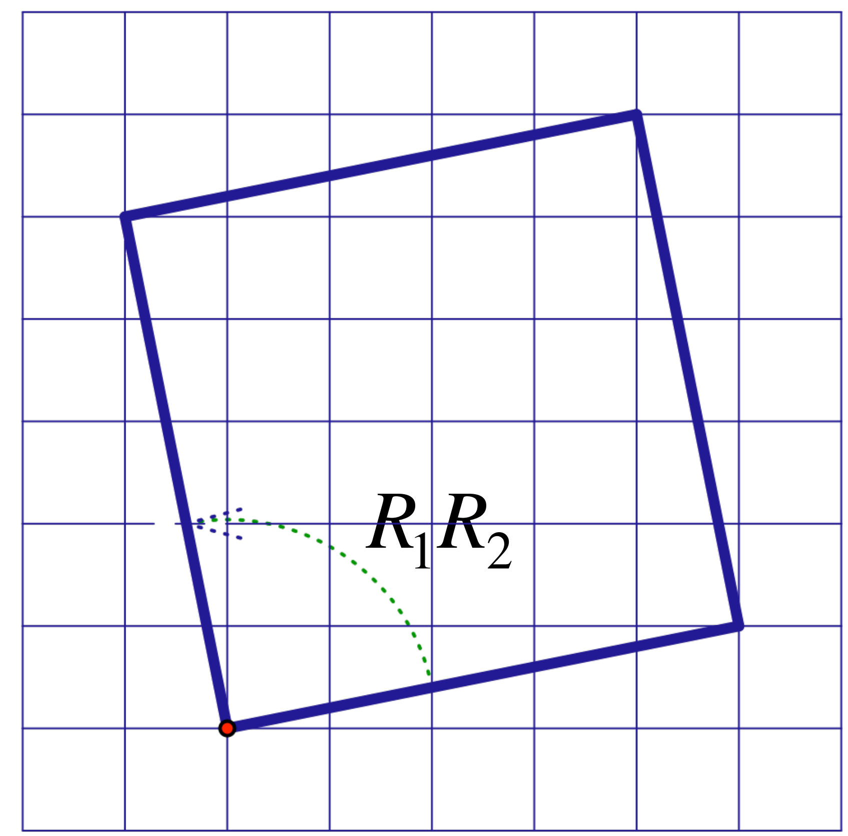
Chiral {4,4} toroids

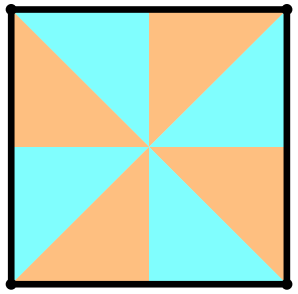
Coxeter 1948



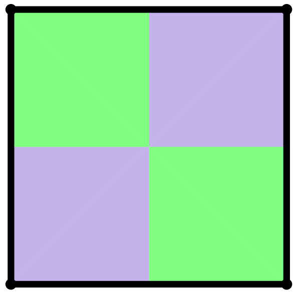
$\mathbf{K} = \langle R_1, R_2 \rangle$

$$\{4,4\}_{(a,b)(-b,a)}$$

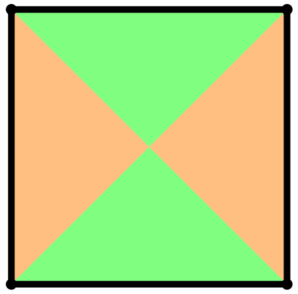




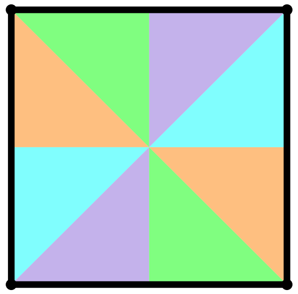
$\mathbf{K} = \langle R_1R_2 \rangle$



$\mathbf{K} = \langle t, R_1 \rangle$



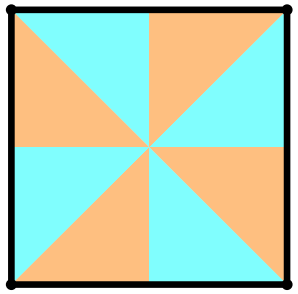
$\mathbf{K} = \langle t, R_2 \rangle$



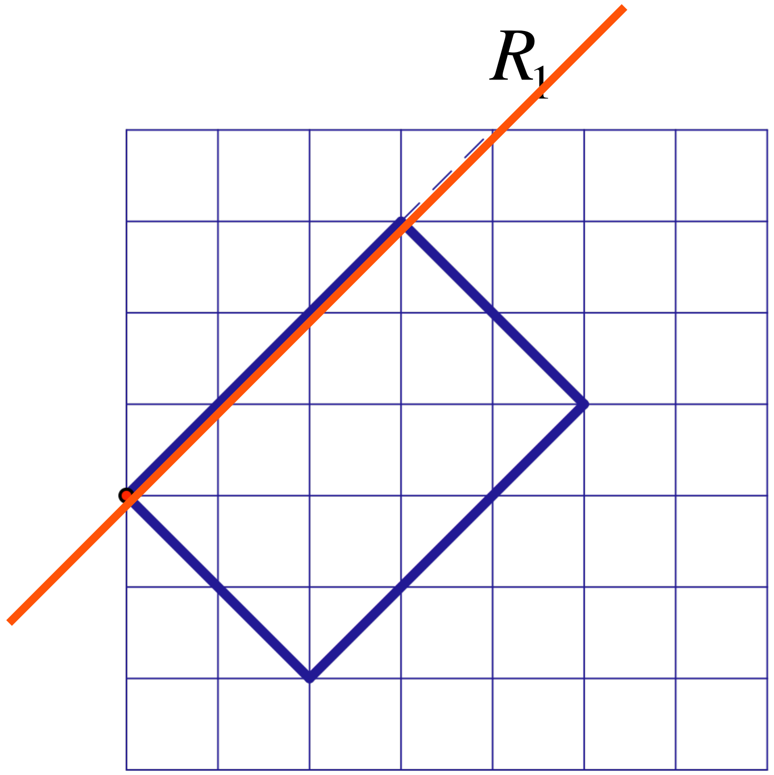
$\mathbf{K} = \langle t \rangle$

Class 2_1. Edge transitive

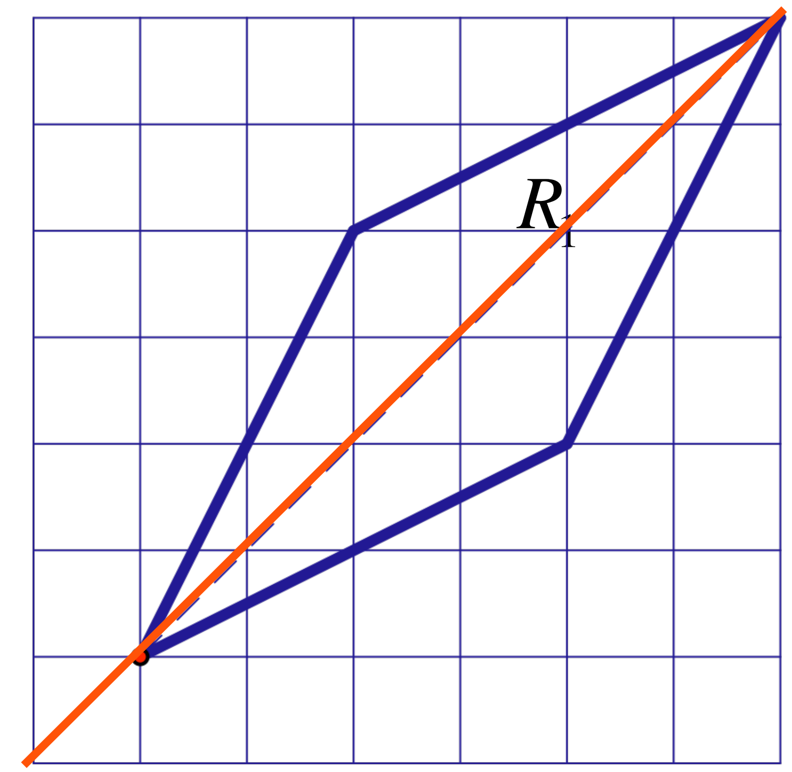
Širán, Tucker, Watkins, 2001



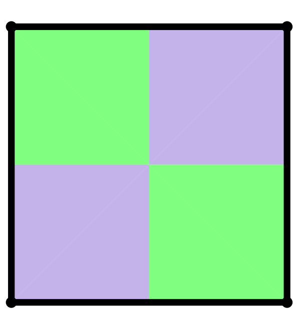
$\mathbf{K} = \langle R_1 R_2 \rangle$



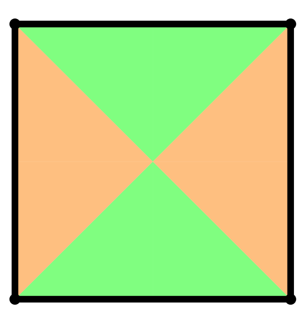
$\{4,4\}_{(a,a)(b,-b)}$



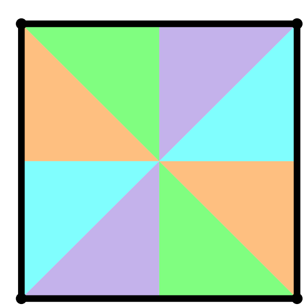
$\{4,4\}_{(a,b)(b,a)}$



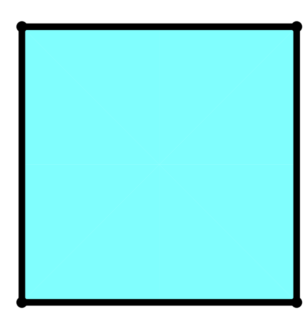
$\mathbf{K} = \langle t, R_1 \rangle$



$\mathbf{K} = \langle t, R_2 \rangle$



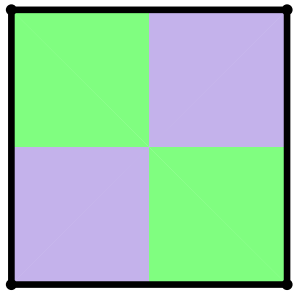
$\mathbf{K} = \langle t \rangle$



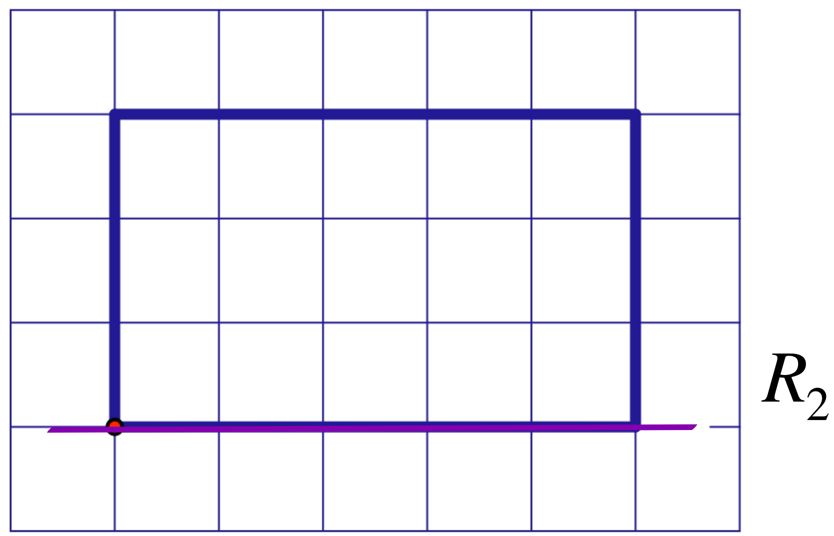
$\mathbf{K} = \langle R_1, R_2 \rangle$

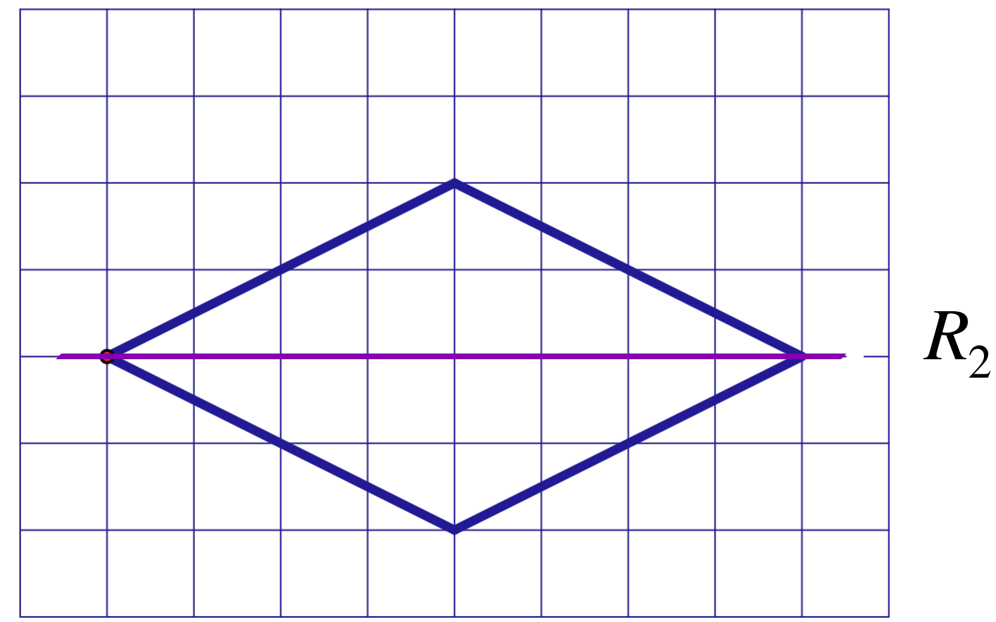
Class 2_{02}. Face & vertex transitive

Duarte, 2007; H., 2007



$K = \langle t, R_1 \rangle$





$\{4,4\}_{(a,0)(0,b)}$,

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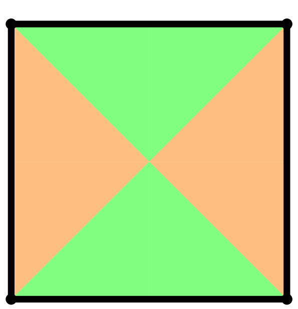
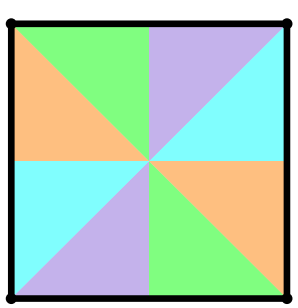
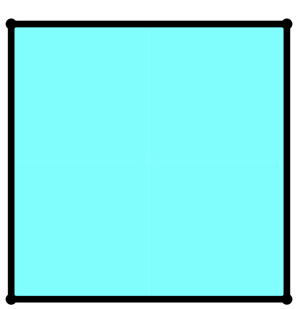


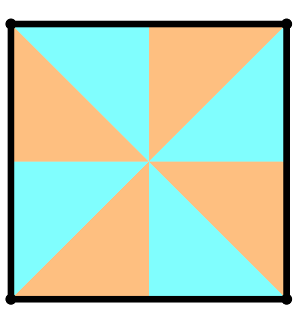
$K = \langle t, R_2 \rangle$



$K = \langle t \rangle$

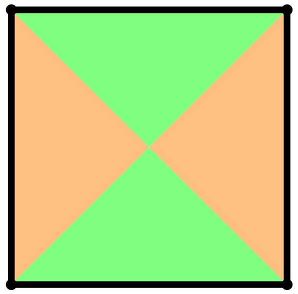


$K = \langle R_1, R_2 \rangle$



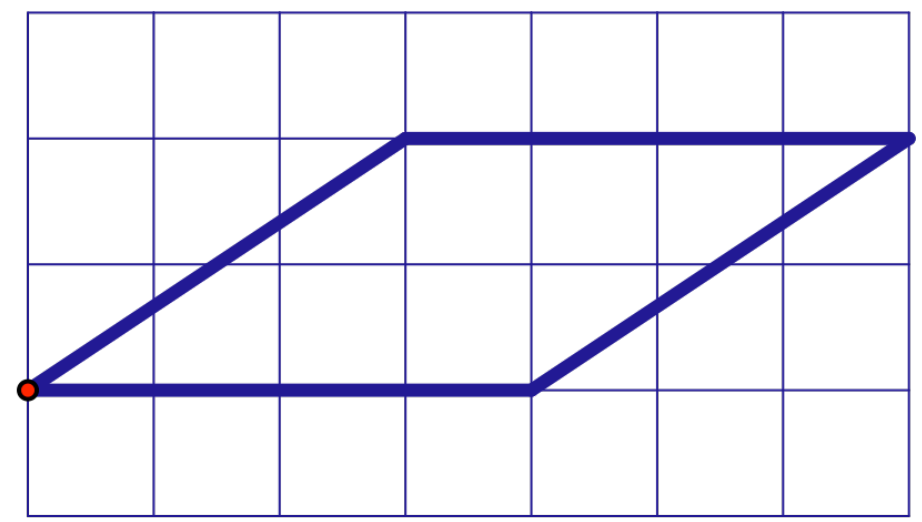
$K = \langle R_1 R_2 \rangle$

Class 4. Face & vertex transitive



K $=\langle t, R_2 \rangle$

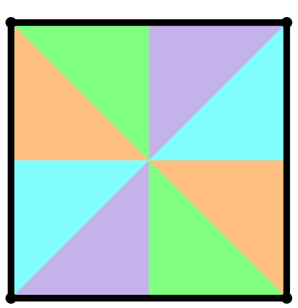
H., Orbanič,
Pellicer, Weiss 2012



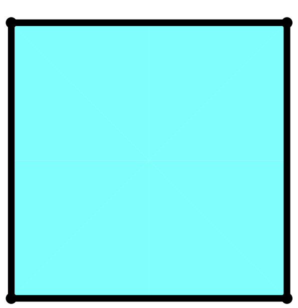
$a > b > 0, \quad c \geq a - b, \quad c \neq 2a \neq 4c$

and if $b | a, c$, then $\frac{c}{b} \nmid 1 \pm \frac{a^2}{b^2}$

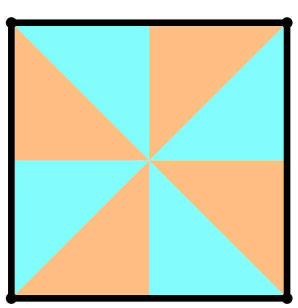
$\{4,4\}_{(a,b)(c,0)}$



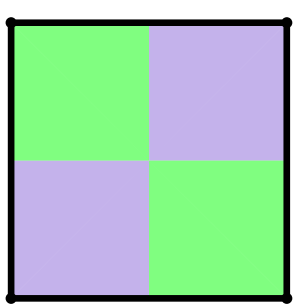
K $=\langle t \rangle$



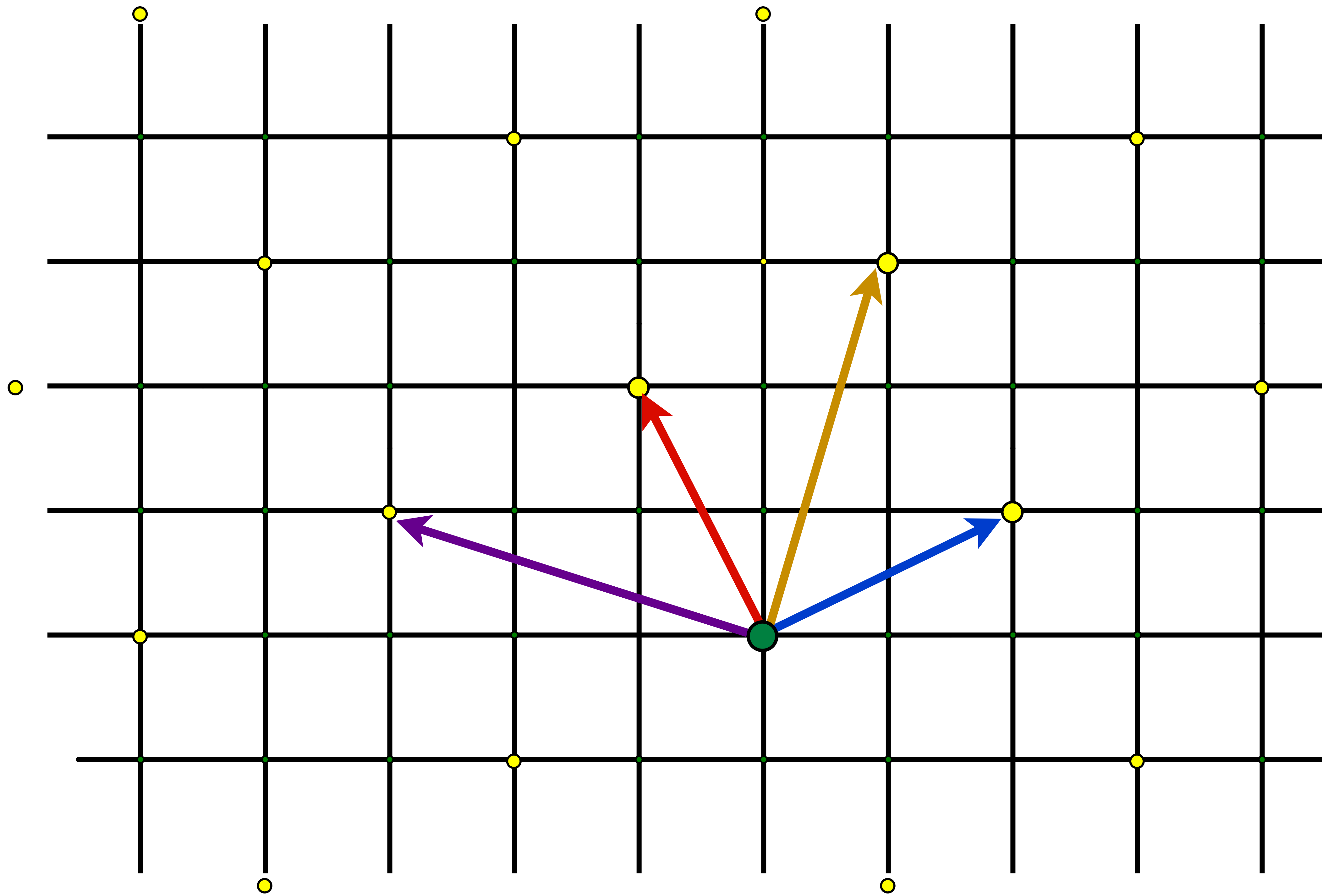
K $=\langle R_1, R_2 \rangle$

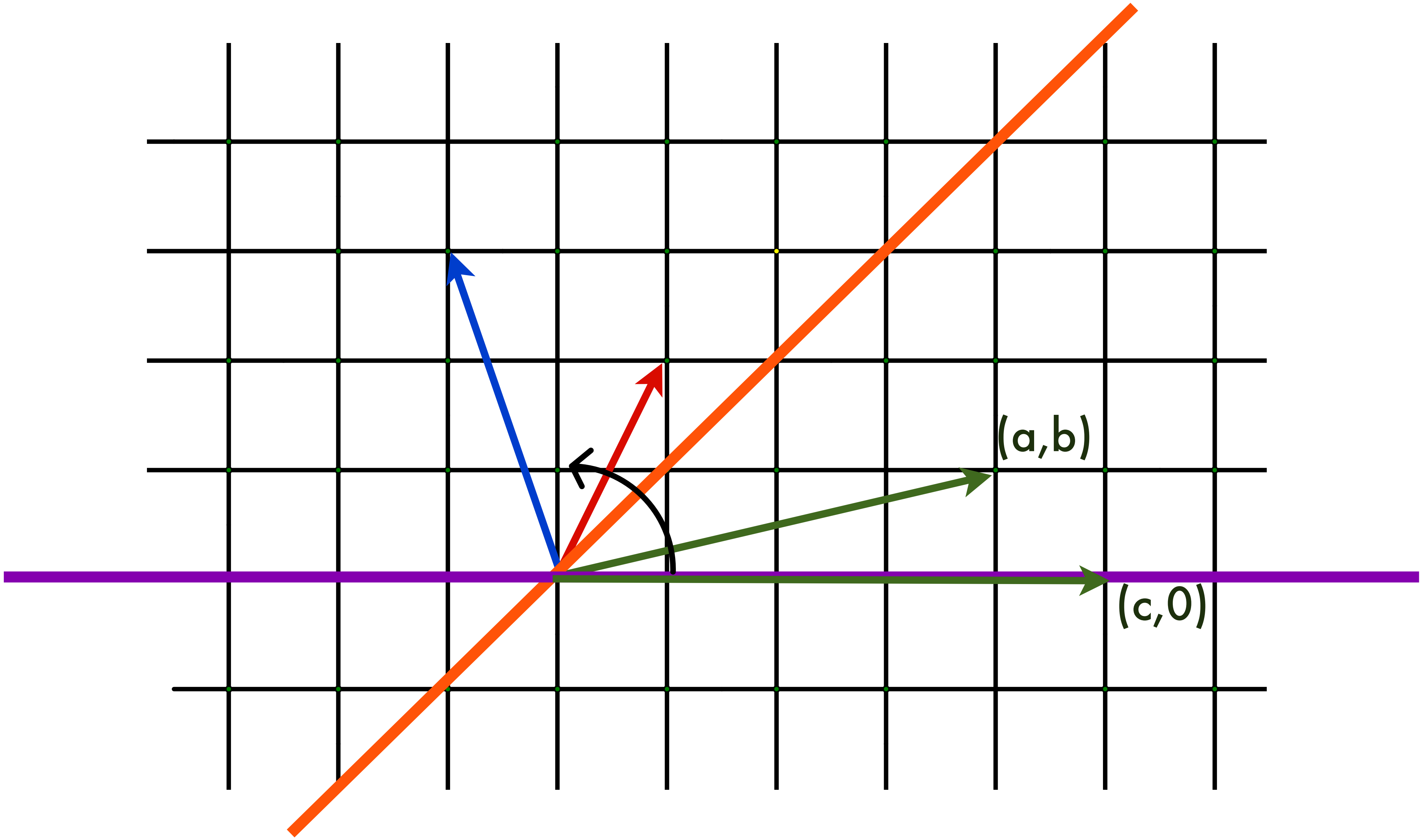


K $=\langle R_1 R_2 \rangle$



K $=\langle t, R_1 \rangle$



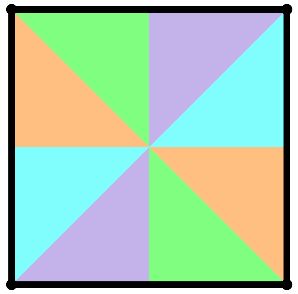


R_1 is a symmetry if and only if $b|a,\ b|c$ & $\frac{c}{b}\left|1 - \frac{a^2}{b^2}\right.$.

R_2 is a symmetry if and only if $c|2a$.

$R_1\ R_2$ is a symmetry if and only if $b|a,\ b|c$ & $\frac{c}{b}\left|1 + \frac{a^2}{b^2}\right.$.

Class 4. Face & vertex transitive



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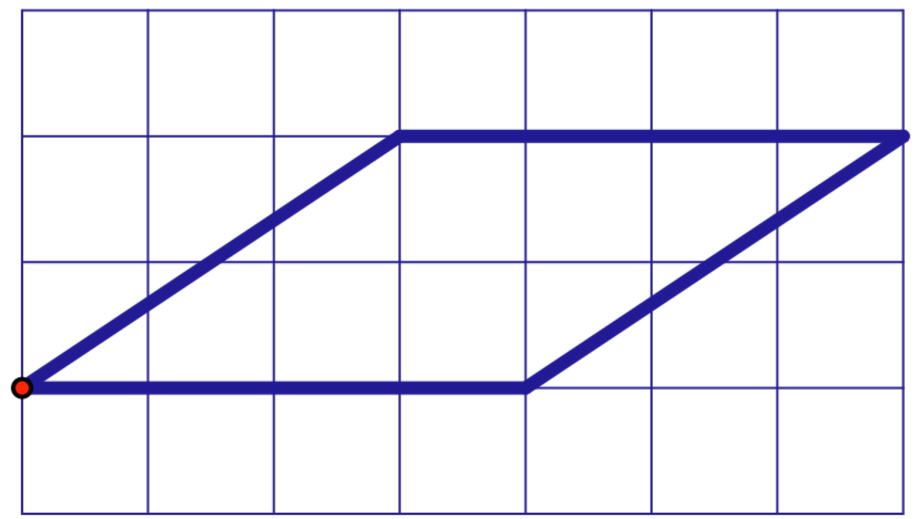

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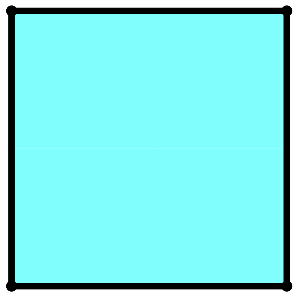
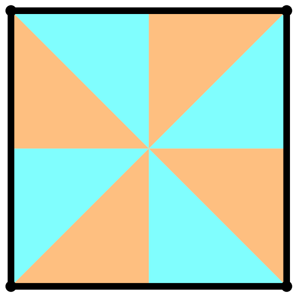
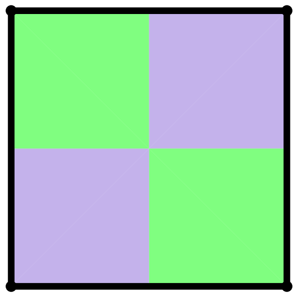


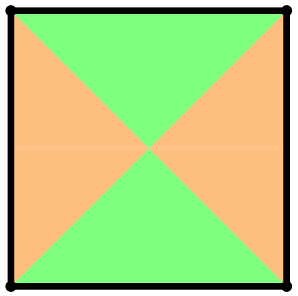
K $= \langle R_1, R_2 \rangle$



K $= \langle R_1 R_2 \rangle$



K $= \langle t, R_1 \rangle$



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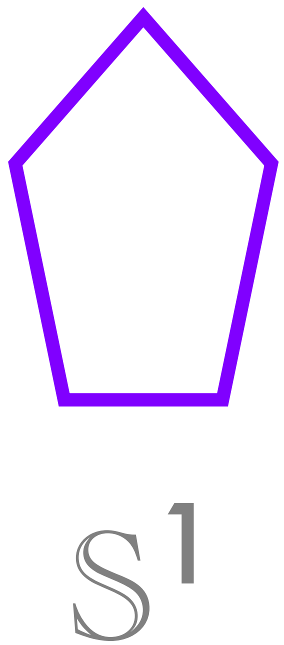
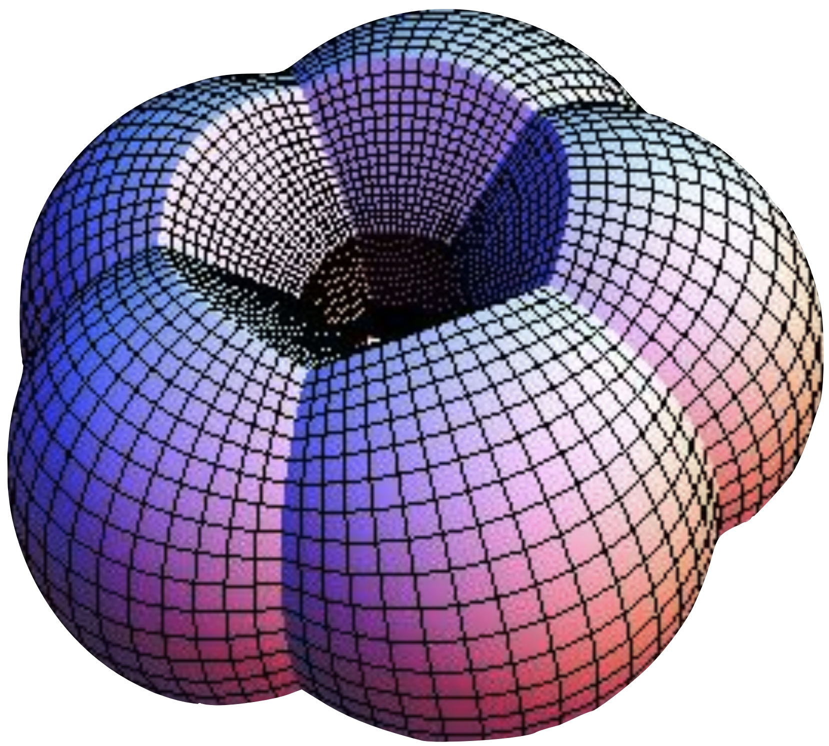
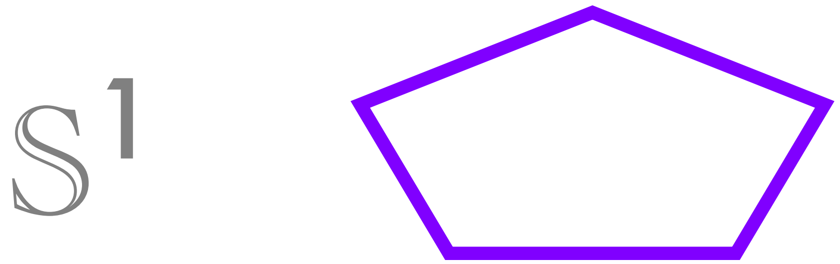
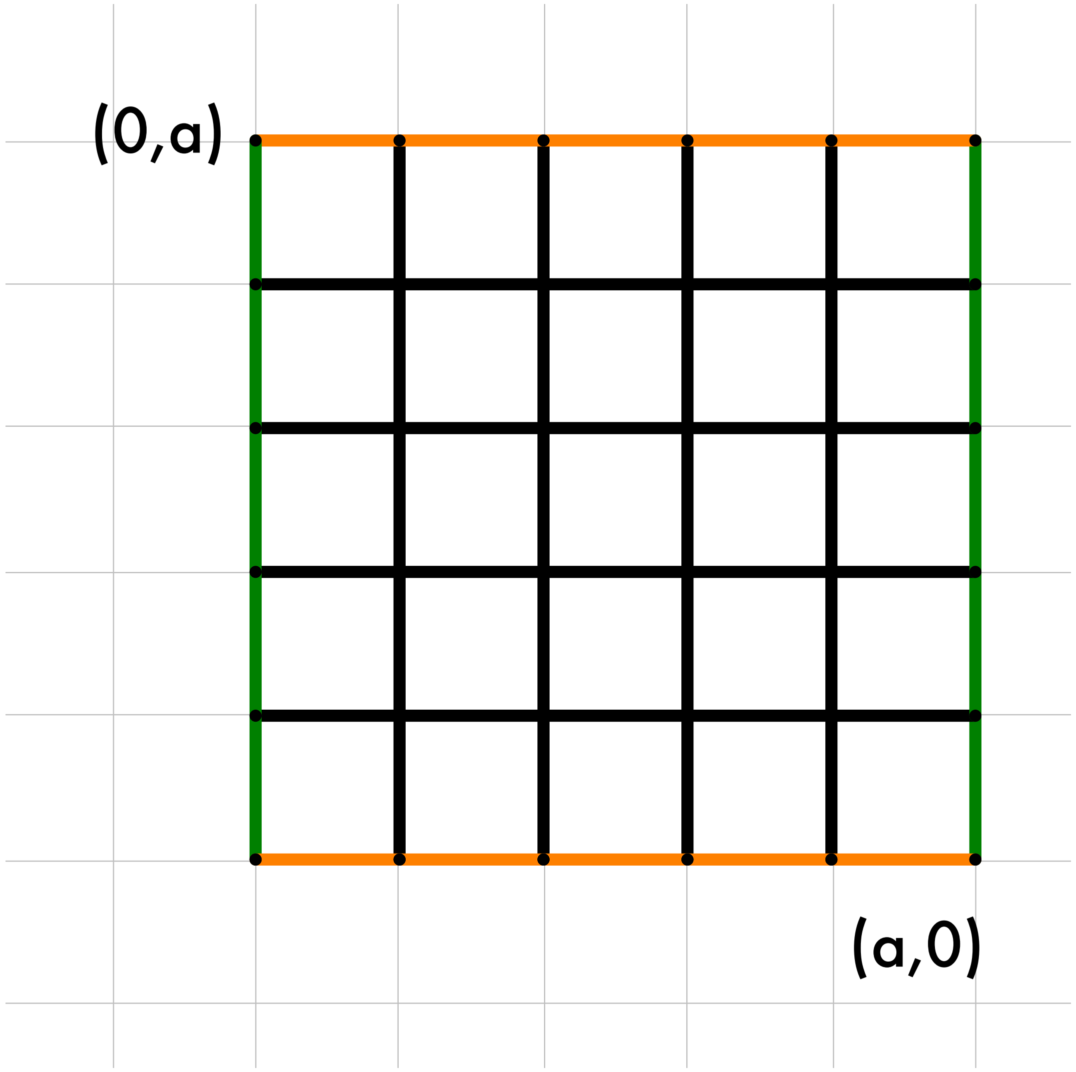
Regular toroids in $\mathbb{R}^4$

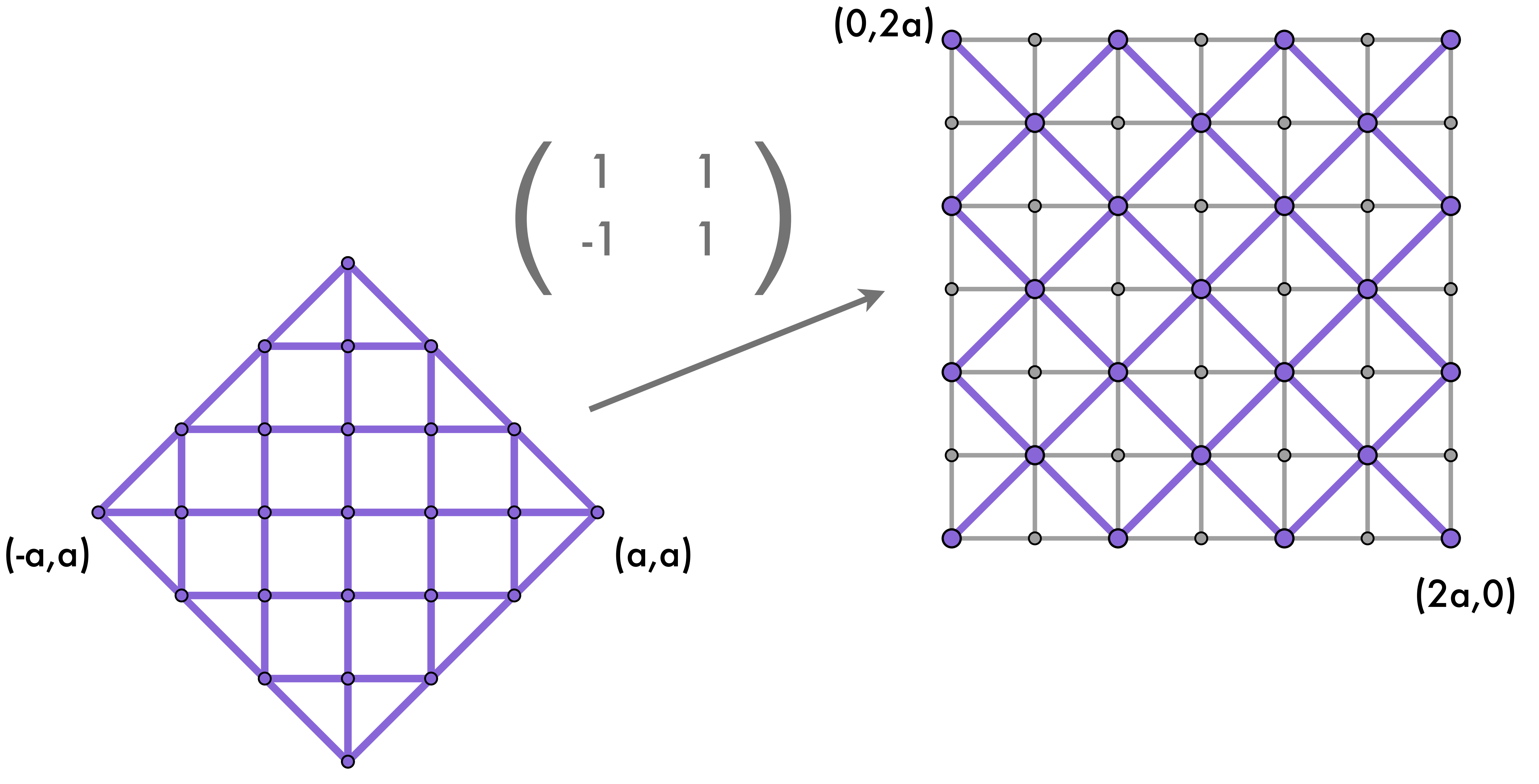






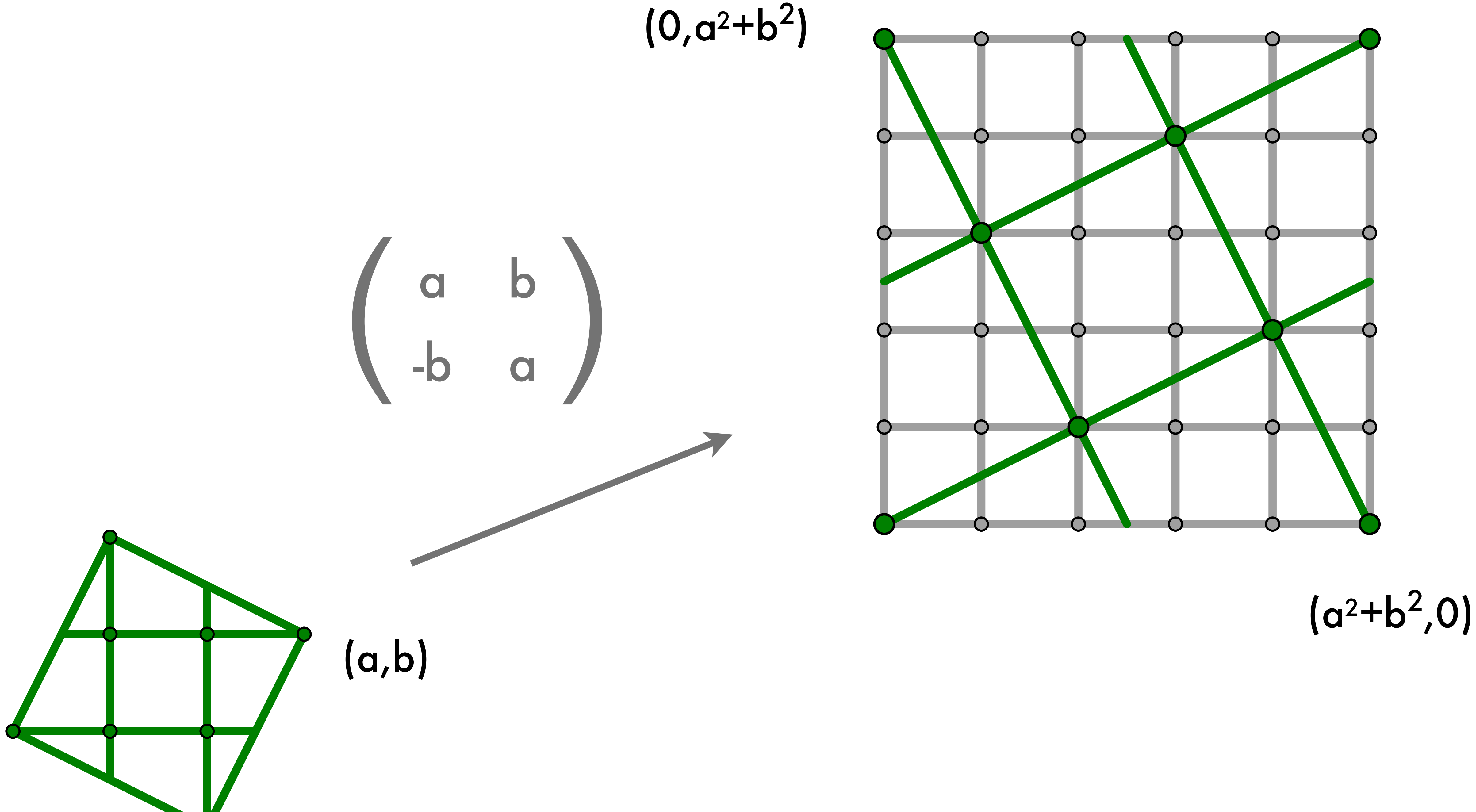


Regular

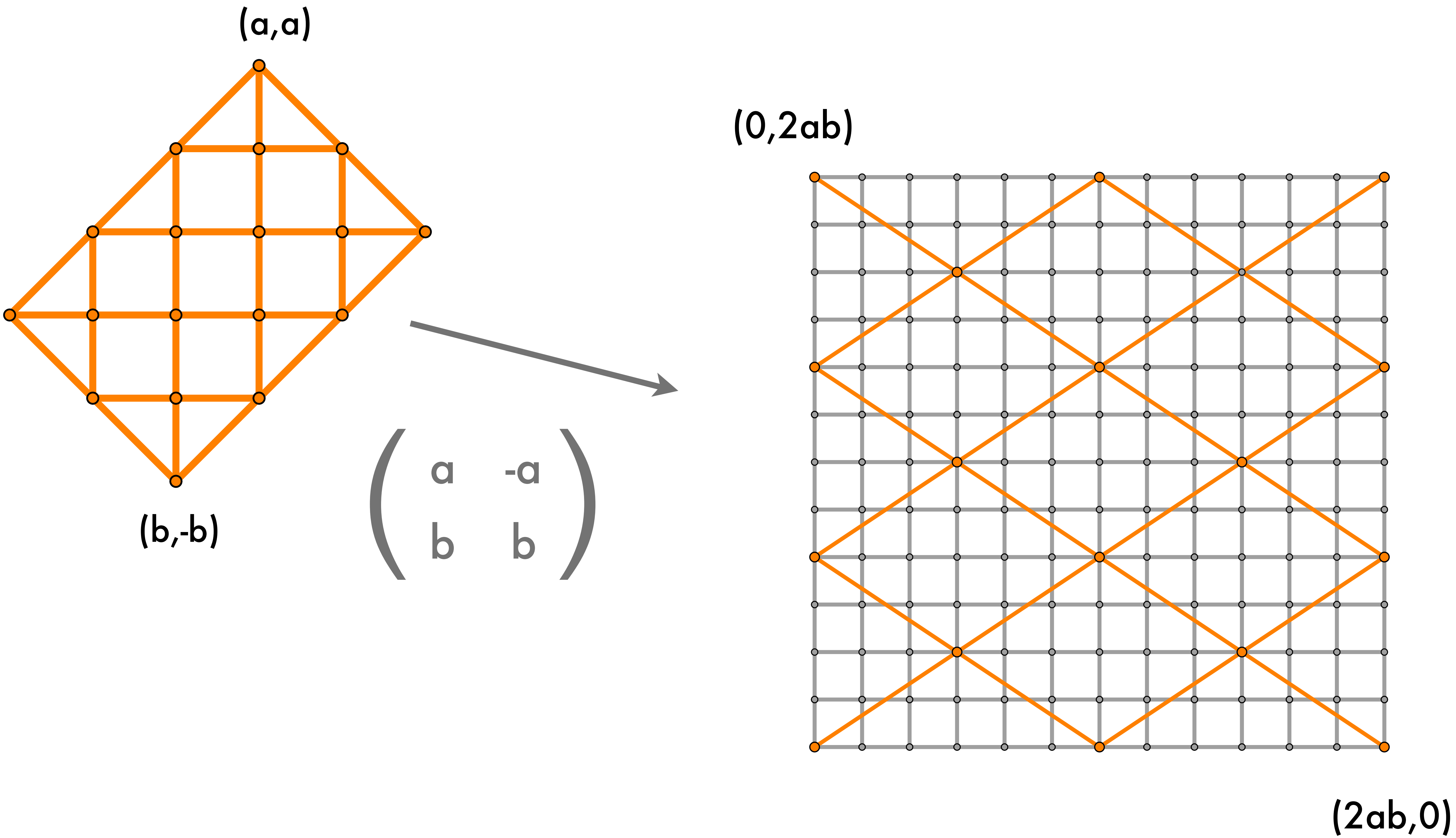


$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

Chiral



Class 2_1



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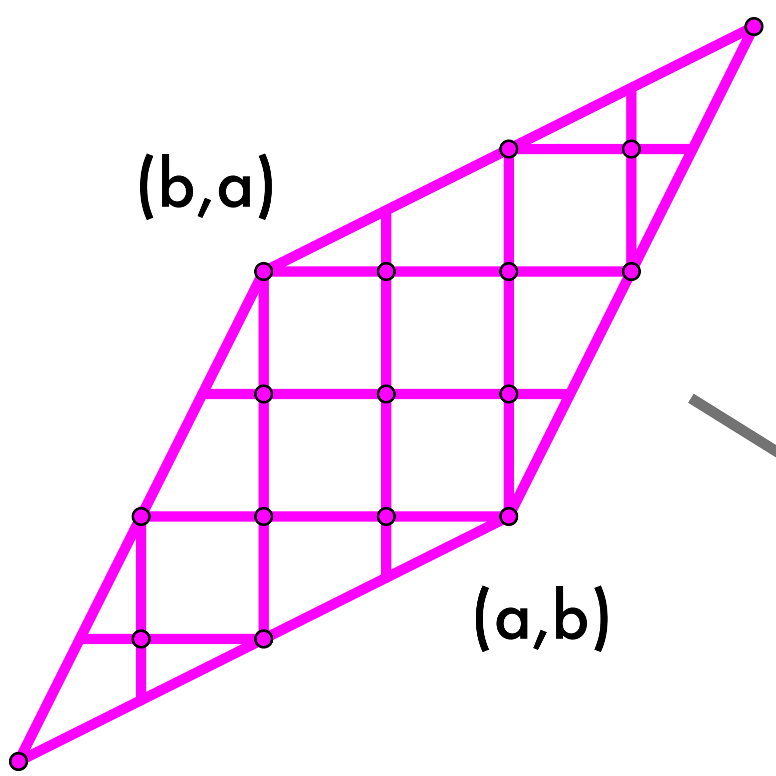
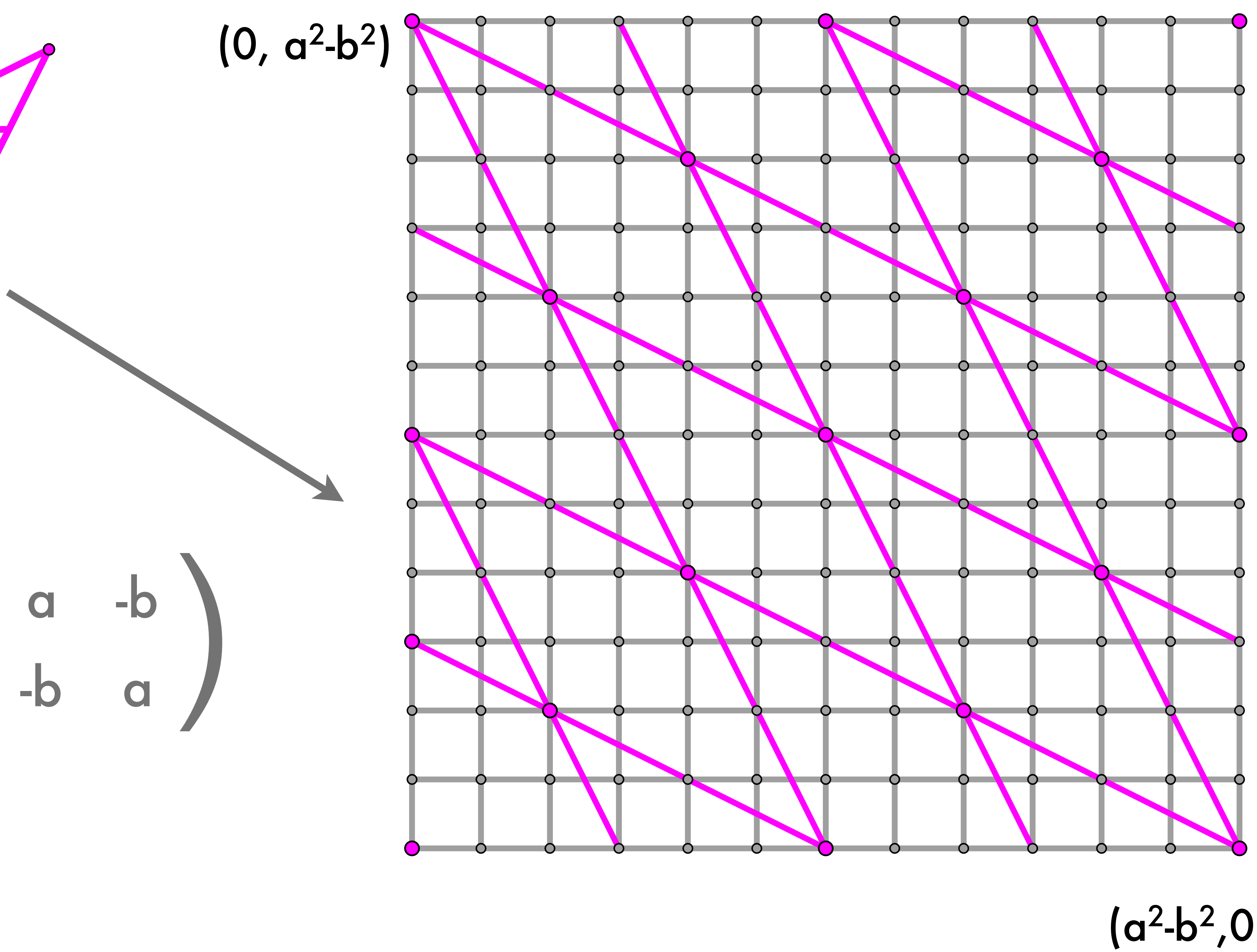
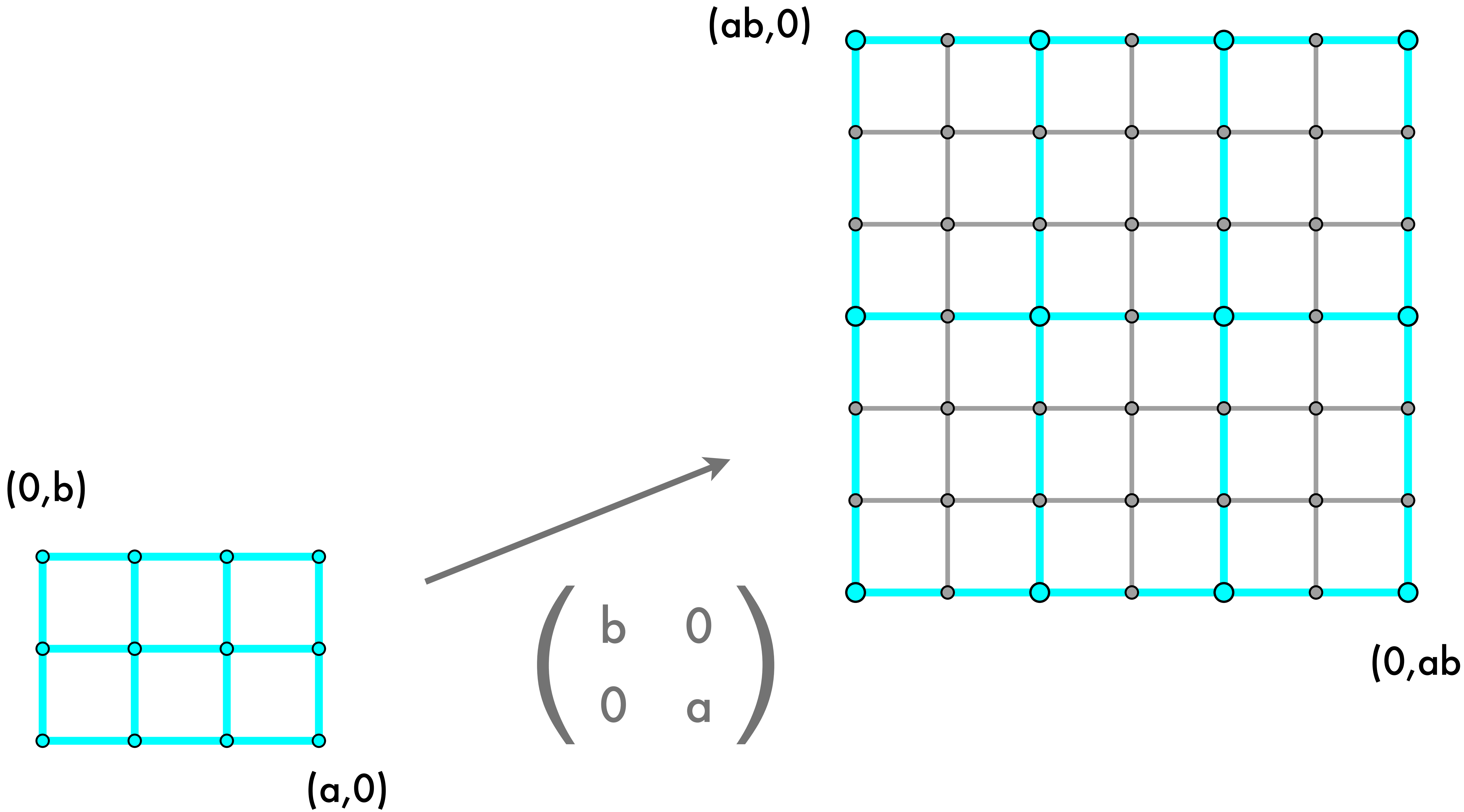


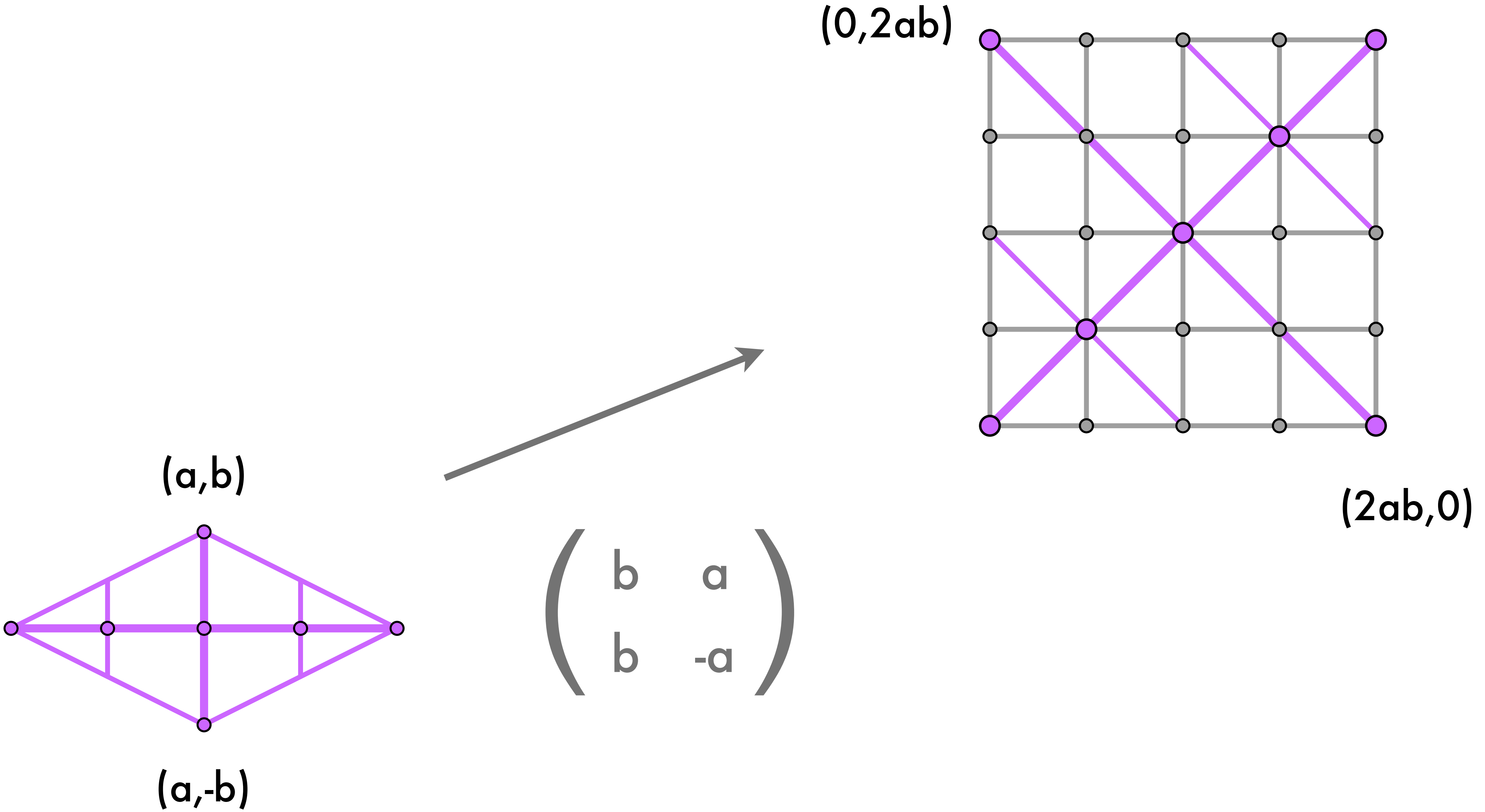
$$\begin{pmatrix} a & -b \\ -b & a \end{pmatrix}$$



Class 2_{02}

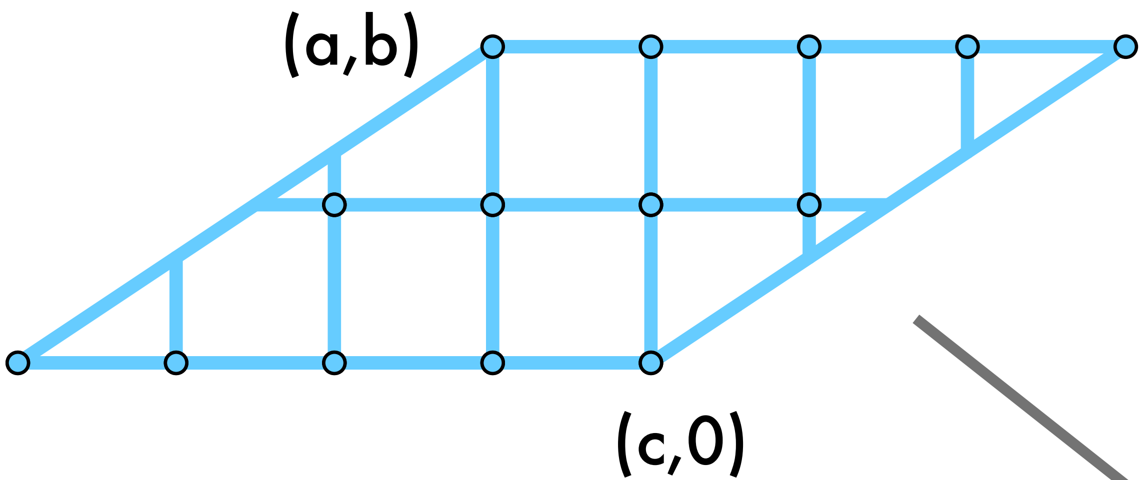


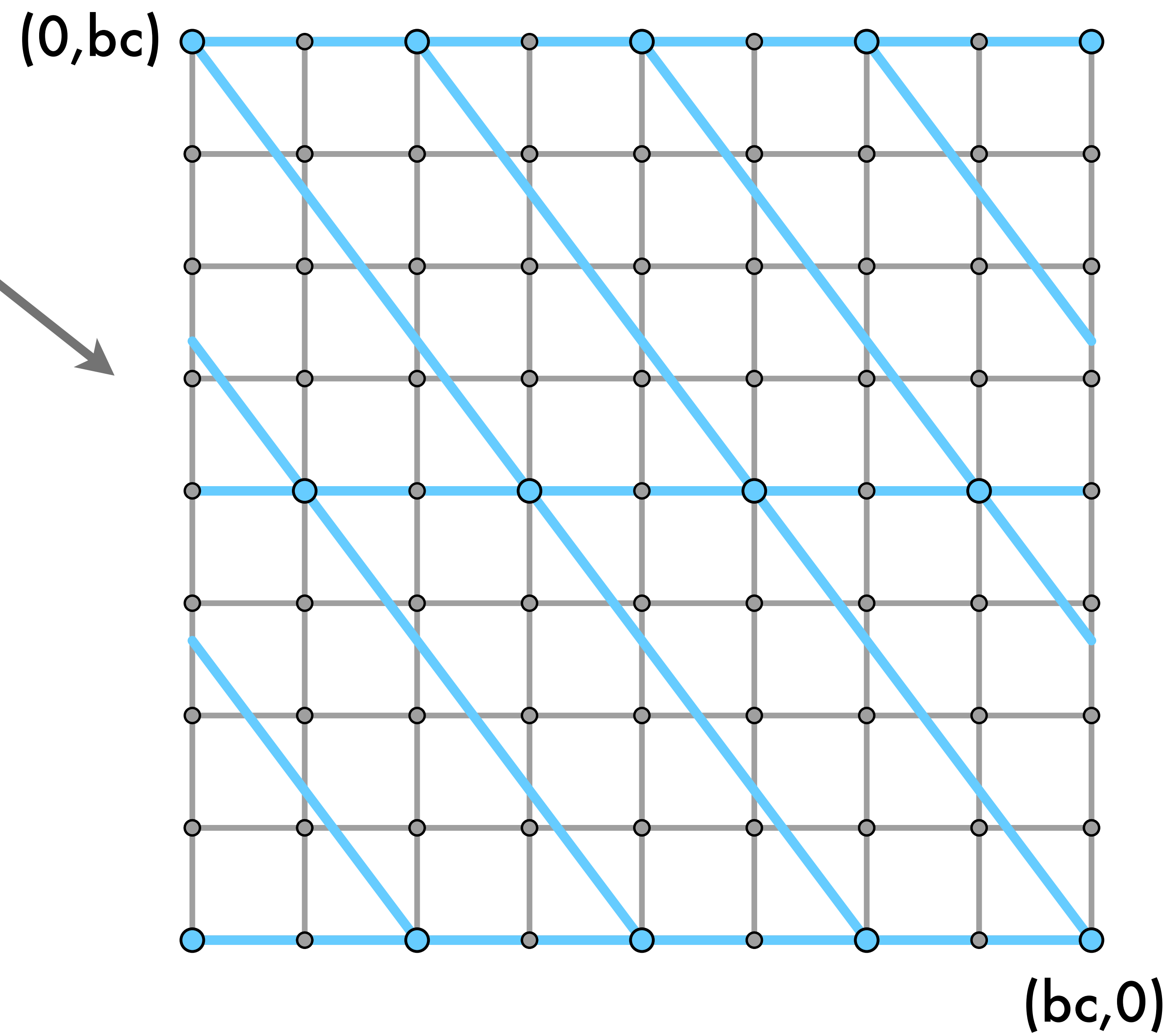
Class 2_{02}



$$\begin{pmatrix} b & a \\ b & -a \end{pmatrix}$$

Class 4

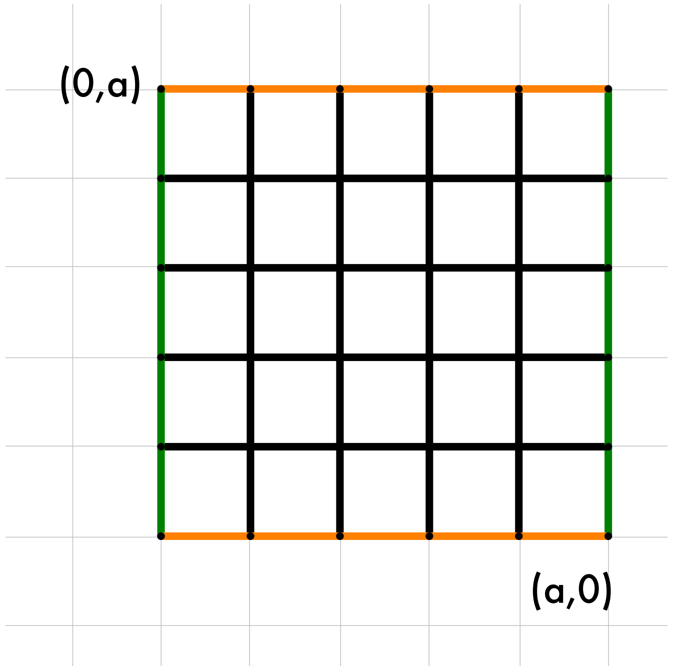




$$\begin{pmatrix} b & -a \\ 0 & c \end{pmatrix}$$

Theorem (Bracho, H., Pellicer)

If you can realise every regular



toroid in some metric space, then you can realise every {4,4} toroid in that space.

Thank you!