Graph-restrictive permutation groups and the PSV Conjecture

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joint work with Luke Morgan

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A Theorem of Tutte

(1947,1959)

Let Γ be a finite connected cubic graph with an arc-transitive group G of automorphisms. Then $|G_v| \leqslant 48$.

Corollary: $|G| \leqslant 48|V\Gamma|$.

Graph-restrictive

Γ a finite connected graph with $G \leqslant \mathrm{Aut}(\Gamma)$ transitive on vertices.

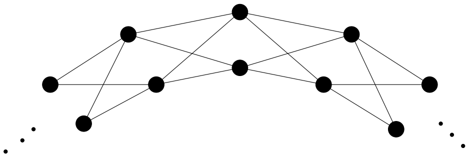
$G_v^{\Gamma(v)}$ is the permutation group induced on $\Gamma(v)$ by G_v.

Given a permutation group L, we say that the pair (Γ, G) is locally L if $G_v^{\Gamma(v)} \cong L$ for all vertices v.

We say that L is graph-restrictive if there is a constant C such that for all locally L pairs (Γ, G), we have that $|G_v| \leqslant C$.

Tutte: C_3 and S_3 are graph-restrictive.

A nonexample



$\mathrm{Aut}(\Gamma) = S_2 \,\mathrm{wr}\, D_{2n}$

$\mathrm{Aut}(\Gamma)_v^{\Gamma(v)} = D_8$

$|\,\mathrm{Aut}(\Gamma)_v| = 2^{n-1}.2$

An equivalent definition

$G_v^{[i]}$ is the kernel of the action of G_v on the set of all vertices at distance at most i from v.

L is graph-restrictive if and only if there is some constant k such that for all locally L pairs (Γ, G) we have $G_v^{[k]} = 1$.

Given an edge $\{v, w\}$, $G_{vw}^{[1]}$ is the kernel of the action of G_{vw} on $\Gamma(v) \cup \Gamma(w)$.

Some graph-restrictive groups

- Any regular group.
- Gardiner (1973): Any transitive subgroup of S_4 other than D_8.
- Sami (2006): D_{2n} for n odd.
- Trofimov, Weiss: any 2-transitive group.
- Verret (2009): Groups L such that $L = \langle L_x, L_y \rangle$ and L_x induces C_p on y^{L_x} for some prime p (p-subregular).

D_{2n}, for n odd, is 2-regular

Primitive groups and generalisations

Let $G \leqslant \mathrm{Sym}(\Omega)$.

- Call G primitive if the only partitions of Ω that it preserves are the trivial ones $\{\Omega\}$ and $\{\{\omega\} \mid \omega \in \Omega\}$.
- Call G quasiprimitive if every nontrivial normal subgroup is transitive.
- Call G semiprimitive if every nontrivial normal subgroup is transitive or semiregular.

Semiprimitive groups

Initially studied by Bereczky and Maróti.

Examples include:

- primitive and quasiprimitive groups;
- regular groups;
- Frobenius groups (that is, all nontrivial elements fix at most one point);
- $\mathrm{GL}(n, p)$ acting on the set of nonzero vectors of $\mathbb{Z}_p^n$.

Weiss Conjecture

Weiss Conjecture (1978): Every primitive group is graph-restrictive.

Weiss (1979): If L is a primitive permutation group of affine type on p^d points for $p \geq 5$, then L is graph-restrictive.

Praeger, Spiga, Verret (2012): Reduced to a problem about simple groups.

Praeger, Pyber, Spiga, Szabó (2012): Weiss conjecture is true if composition factors in G have bounded rank.

What is the correct setting?

Praeger Conjecture: Every quasiprimitive group is graph-restrictive.

Potočnik, Spiga, Verret (2012): If a transitive group is graph restrictive then it is semiprimitive.

PSV conjecture: A transitive group is graph-restrictive if and only if it is semiprimitive.

D_8 is not semiprimitive as it contains a normal intransitive subgroup isomorphic to C_2^2.

Spiga, Verret (2014): An intransitive group is graph-restrictive if and only if it is semiregular.

Variation on Thompson-Wielandt

Spiga (2012): If (Γ, G) is a locally semiprimitive pair and $\{v, w\}$ is an edge such that $G_{vw}^{[1]} \neq 1$ then $G_{vw}^{[1]}$ is a p-group.

Regular nilpotent normal subgroups

Giudici and Morgan

Let L be a semiprimitive group with a regular normal nilpotent subgroup K.

(A group is nilpotent if and only if it is the direct product of its Sylow subgroups.)

Theorem Every transitive normal subgroup contains K, and every semiregular normal subgroup is contained in K.

Theorem Let (Γ, G) be a locally L pair with $|K|$ coprime to 6. Then $G_{vw}^{[1]} = 1$ and so L is graph-restrictive.

Semiprimitive groups of this type include:

- affine primitive groups on p^n points for $p \geq 5$;
- Frobenius groups of degree coprime to 6;
- $P \rtimes C_2$ with P a regular abelian p-group for $p \geq 5$ and C_2 acting by inversion;
- $p_+^{1+2m} \rtimes \mathrm{Sp}(2m, q)$ with $p \geq 5$.
- $V = \mathrm{GF}(q)^n$ and $G = (V \oplus V \oplus \cdots \oplus V) \rtimes \mathrm{GL}(V)$

More detailed information

Also give detailed information about what a counterexample with order not coprime to 6 must look like.

Theorem Let (Γ, G) be a locally L pair where L is semiprimitive with a regular normal nilpotent subgroup K and suppose that $G_{vw}^{[1]} \neq 1$. Then L contains normal subgroups F and J such that $F < K < J$ and either

- $G_{xy}^{[1]}$ is a 2-group and $J/F \cong S_3 \times \cdots \times S_3$, or
- $G_{xy}^{[1]}$ is a 3-group and $J/F \cong A_4 \times \cdots \times A_4$.

Small groups

Potočnik, Spiga and Verret looked at all transitive groups of degree at most 13. The only ones whose status at the time were unknown were:

- $S_3 \operatorname{wr} S_2$ on 9 points (primitive)
- $3^2 \rtimes 2$ on 9 points (imprimitive)
- Sym(5) on 10 points (primitive)
- Sym(4) on 12 points (imprimitive)

A class of Frobenius groups

Let L be the Frobenius group $C_3^n \rtimes C_2$ acting on 3^n points with $n \geq 1$.

Theorem If (Γ, G) is a locally L pair then $G_v^{[4]} = 1$ and so L is graph restrictive.

Tutte's Theorem is the case $n = 1$.