Orientably-Regular Embeddings of Graphs of Order Prime-Cube

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Surfaces and Embeddings

Surface S: closed, connected 2-manifold;

Classification of Surfaces:
(i) Orientable Surfaces: S_g, $g = 0, 1, 2, \cdots$,
$v + f - e = 2 - 2g$

(ii) Nonorientable Surfaces: N_k, $k = 0, 1, 2, \cdots$,
$v + f - e = 2 - k$

Embeddings of a graph X in the surface is a continuous one-to-one function $i : X \to S$.

2-cell Embeddings: each region is homemorphic to an open disk.

Topological Map $\mathcal{M}$: a 2-cell embedding of a graph into a surface. The embedded graph X is called the *underlying graph* of the map.

Automorphism of a map $\mathcal{M}$: an automorphism of the underlying graph X which can be extended to self-homeomorphism of the surface.

Orientation-Preserving Automorphism of an orientably map $\mathcal{M}$: an automorphism of Preserving Orientation of the map

Automorphism group $\mathrm{Aut}(\mathcal{M})$: all the automorphisms of the map $\mathcal{M}$.

Orientation-preserving automorphisms group $\mathrm{Aut}^{+}\mathcal{M}$ of $\mathcal{M}$: all the oientation-preserving automorphism.

Flag: incident vertex-edge-face triple

Arc: incident vertex-edge pair

Remark: $\mathrm{Aut}(\mathcal{M})$ acts semi-regularly on the flags of X.

Remark: $\mathrm{Aut}^+(\mathcal{M})$ acts semi-regularly on the arcs of X.

Regularity of Maps

Regular Map: $\mathrm{Aut}(\mathcal{M})$ acts regularly on the flags.

Orientably Regular Map: $\mathrm{Aut}^{+}(\mathcal{M})$ acts regularly on the arcs.

Reflexible Map: Orientably Regular, admiting orientation-reversing automorphisms

Chiral Map: Orientably Regular, without any orientation-reversing automorphisms

Regular Map
=Nonorientably Regular Map
$\cup$ Reflexible Orientably Regular Map

Orientably Regular Map
=Reflexible Orientably Regular Map
$\cup$ Chiral Orientably Regular Map

Combinatorial Orientably Map:

graph $X = (V, D)$, with vertex set $V = V(X)$, dart (arc) set $D = D(X)$.

arc-reversing involution L: interchanging the two arcs underlying every given edge.

rotation R*:* cyclically permutes the arcs initiated at v for each vertex $v \in V(X)$.

Map $\mathcal{M}$ *with underlying graph* X*:*
the triple $\mathcal{M} = \mathcal{M}(X;\ R,\ L)$.

Remarks:

Monodromy group $\mathrm{Mon}(\mathcal{M}) := \langle R, L \rangle$ acts transitively on D.

Given two maps

$$\mathcal{M}_1 = \mathcal{M}(X_1;\ R_1,\ L_1),\ \mathcal{M}_2 = \mathcal{M}(X_2;\ R_2\, L_2),$$

Map isomorphism: bijection $\phi : D(X_1) \to D(X_2)$ such that

$$L_1\phi = \phi L_2, R_1\phi = \phi R_2$$

Automorphism ϕ of $\mathcal{M}$: if $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}$;

Automorphism group: $\mathrm{Aut}(\mathcal{M})$

Algebraic Orientably Maps:

Orientably Regular Map:

$G = \mathrm{Aut}(\mathcal{M}) = \langle r, l \rangle \cong \mathrm{Mon}(\mathcal{M}) = \langle R, L \rangle$

$\mathcal{M} = \mathcal{M}(G; r, l)$

$D = G$, $\mathrm{Mon}(\mathcal{M}) = L(G)$, $\mathrm{Aut}(\mathcal{M}) = R(G)$

the orbits of $\langle r \rangle$, $\langle l \rangle$ and $\langle rl \rangle$ are vertices, edges and faces, with the natural inclusion relation

ORM without multiple edges

A regular map with multiple edges projects onto another one with a simple underlying graph that has the same set of vertices and the same adjacency relation.

Regular maps with multiple edges can be described as some "extensions" of regular embeddings of simple graphs.

$G = \langle r, \ell \rangle \to$

$\overline{G} = G/K, \quad K = \langle r \rangle_G$—core of $\langle r \rangle$ in G

To determine the ORM with multiple edges from an ORM of a simple graph is essentially a group cyclic extension problem.

Here we just consider the ORM without multiple edges

2. Regular maps of given order

Set v=order of graphs

ORM with given order $v \Leftrightarrow$ ORM with given graphs

because one may pick up the symmetric graphs of order v with a arc-regular subgroup.

1. $v = p =$a prime:

$$G = \langle a, b \mid a^q = b^s = 1, a^b = a^t \rangle,$$

$$\mathcal{M} = \mathcal{M}(G; b^e, (b^{\frac{s}{2}})^a), \quad e \in \mathbb{Z}_s^*.$$

S.F. Du, J.H. Kwak and R. Nedela, Regular embeddings of complete multipartite graphs, *EJC* 26(2005), 437–452

2. $v = pq$=a product of two primes

ORM: S.F. Du, J.H. Kwak and R. Nedela, A classification of regular embeddings of graphs of order a product of two primes, *JAC* 19(2004), 123–141.

NORM: S.F.Du, J.H. Kwak and F.R. Wang, published in two papers: DM and Sciences in China.

3. $v = p^3$:

Motivations:

(a) To understand permutation groups of degree p^3 in more details (subgroup structure) is still a difficult problem.

(b) Classification for symmetric graphs of order p^3 is still open.

(c) Complete classification for semi-symmetric symmetric graphs order $2p^3$ is still open,

only partial results are given, that is $\mathrm{Aut}(X)$ acts infaithfully on one bipart, see

(i) L. Wang, S.F. Du, X.W. Li, A Class of Semisymmetric Graphs, *AMC*, 7 (2014) 40 C53

(ii) L. Wang , S. F. Du, SEMISYMMETRIC GRAPHS OF ORDER $2p^3$, *EJC*, 36 (2014) 393 C405

(iii) S.F. Du, L. Wang, A Classification of Semisymmetric Graphs of Order $2p^3$: Unfaithful Case, *JAC*, DOI 10.1007/s10801 -014-0536-3 (28 pages)

$\mathrm{Aut}(X)$ acts faithfully on each bipart: that is a hard part for this work.

(d) For ORM of order p^3, we may do that, because we do have a particular subgroup of degree p^3, that is $\mathrm{Aut}(\mathcal{M}) = \langle r, \ell \rangle$, which is an arc-regular subgroup of the graph.

(e) Many recent results on ORM can help us to do this work.

3. ORM of order p^3

Notation:

Γ=a connected simple graph of order p^3 where p is prime and of valency n

$\mathcal{M}$=an ORM of $\mathcal{G}$

$G = \langle r, \ell \rangle$=the orientation preserving group of $\mathcal{M}$

$\ell^2 = 1$, $\langle r \rangle = G_v$ for a vertex v in $V(\Gamma)$.

P =a Sylow subgroup of G

N=a minimal normal subgroup of G

B= the orbits of N on the vertices

K= be the kernel of G acting on **B** and $\overline{G} = G/K$.

Theorem

(1) $|P| = p^3, p^4$ *or* p^5.

(2) $G = P \rtimes \langle r^m \rangle$ *where* $m = |\langle r \rangle \cap P|$.

(3) $N = \mathbb{Z}_p^k$, $k = 1, 2, 3$, *and either*

  (3.1) N *is transitive on* V *and* G *is a primitive affine group; or*

  (3.2) N *induce a blocks of length* p *such that* $N \cong \mathbb{Z}_p \leq Z(P)$ *and either* $K \cong \mathbb{Z}_p \rtimes \mathbb{Z}_t$ *for some* $t \in Z_p^*$; *or* $K \cong \mathbb{Z}_p^2$.

Remark: From $G = P \rtimes \langle r^m \rangle$, we need to

study the split cyclic extension of P by $\mathbb{Z}_{n_1}$ where $n = mn_1$ for $m = P \cap \langle r \rangle$ and $m = 1, p, p^2$, $(n_1, p) = 1$.

$\Longleftrightarrow$

to determine the congugacy classes of cyclic subgroups of order prime to p in $\mathrm{Aut}(P)$, noting that $|P| = p^3, p^4$ or p^5

3.2 $|P| = p^5$

This case is quite complicated. Fortunately, it becames more easy, because we may employ many known results !

$|P| = p^5 \Longrightarrow \Gamma$ is a p-partite graph such that any two connected biparts is complete bipartite graph.

Recalling some known results:

$K_m[nK_1]$ =the complete multipartite graph with m parts, while each part contains n vertices.

(i) $m = 1$: Complete graphs:

ORM:

N.L. Biggs, Classification of complete maps on orientable surfaces, *Rend. Mat.* (6) **4** (1971), 132-138.
L.D. James and G.A. Jones, Regular orientable imbeddings of complete graphs, *J. Combin. Theory Ser. B* **39** (1985), 353–367.

NORM:

S. E. Wilson, Cantankerous maps and rotary embeddings of K_n, *JCTB* **47** (1989), 262–273.

(ii) $m = 2$: Complete bipartite graphs $K_2 nK_1 = K_{n,n}$:

ORM:

Survey paper: G.A. Jones, Maps on surfaces and Galois groups, *Math. Slovaca* **47** (1997), 1-33.

$n = p^k$, p is odd prime:
G.A. Jones, R. Nedela and M. Škoviera, Regular embeddings of $K_{n,n}$ where n is an odd prime power, *EJC* **28**(2007), 1863-1875.

$n = 2^k$,
S.F. Du, G.A.Jones, J.H. Kwak, R. Nedela and M. Škoviera, Regular embeddings of $K_{n,n}$ where n is a power of 2. I: Metacyclic case, *EJC* **28** (2007), 1595-1608.

S.F. Du, G.A.Jones, J.H. Kwak, R. Nedela and M. Škoviera, Regular embeddings of Kn,n where n is a power of 2. II: Nonmetacyclic case, *EJC* 31(7), 1946-1956. 2010.

Any n:

G.A. Jones, Regular embeddings of complete bipartite graphs: classification and enumeration, *Proc. London Math. Soc.* **101**(2010), 427-453.

Other partial results:

General approach: R. Nedela, M. Škoviera and A. Zlatoš, Regular embeddings of complete bipartite graphs, *DM* **258**(2002) 379-381.

$n = pq$ J.H. Kwak and Y.S. Kwon, Regular orientable embeddings of complete bipartite graphs, *JGT* **50**(2005), 105-122.

Reflexible maps:, J. H. Kwak and Y. S. Kwon, Classification of reflexible regular embeddings and self-Petrie dual regular embeddings of complete bipartite graphs, *DM* **308**(2008) 2156-2166.

$(n, \phi(n)) = 1$*:* G.A. Jones, R. Nedela and M. Škoviera, G. A. Jones, R. Nedela and M. Škoviera, Complete bipartite graphs with a unique regular embedding, *JCTB* **98**(2008), 241-248.

NORM:

J.H.Kwak and Y.S.Kwon, Classification of nonorientable regular embeddings complete bipartite graphs, *JCTB* **101(2011) 191-205.**

(iii). $m \geq 3$: Complete multipartite graphs $K_{m[n]}$:

$n = p$: S. F. Du, J. H. Kwak, R. Nedela, Regular embeddings of complete multipartite graphs, *EJC* **26**(2005), 505-519.

$m \geq 3$ and $n \geq 2$:

S.F.Du and J.Y.Zhang, A Classification of orientably-regular embeddings of complete multipartite graphs, *EJC*, 36(2014), 437-452.

J.Y.Zhang and S.F.Du, On the orientable regular embeddings of complete multipartite graphs, *EJC* 33(2012), 1303–1312.

General question:

For any connected graph X of order m, let $X[nK_1]$ be the m-partite graph, while each part contains n vertices and the block graph induced by the partition is isomorphic to X. Suppose that X has a RM. Classify the RM of $X[nK_1]$.

X is of prime order:

Y.H.Zhu and S.F.Du, Orientably-regular embeddings of a class of multipartite graphs, to appear in *Science in China*, 2014.

This paper depends heavily on classification of ORM of $K_m[nK_1]$ mentioned as above.

Theorem

Suppose that $|P| = p^5$. Then G, $\mathcal{M}$ and the genus g are given by

(1) $p = 2,\ n = 4:$

$$G_1 \cong \langle a, b, x | a^4 = b^4 = x^2 = 1, [a, b] = 1, a^x = b \rangle,$$

$$\mathcal{M}_1 = \mathcal{M}(G_1; a, x), \quad g = 3.$$

(2) $p = 2,\ n = 4:$

$$G_2 \cong \langle a, b, x | a^4 = b^4 = x^2 = 1, [b, a] = a^2 b^2,$$

$$[a^2, b] = [b^2, a] = 1, a^x = b \rangle,$$

$$\mathcal{M}_2 = \mathcal{M}(G_2; a, x), \quad g = 1.$$

(3) $p = 3,\ n = 18$:

$$G_3 \cong \langle a, b | a^{18} = b^2 = c^{27} = 1, c = a^9 b, c^a = c^2 \rangle,$$

$$\mathcal{M}_3(j) = \mathcal{M}(G_3; a^j, b) \text{ where } j \in \mathbb{Z}_{18}^*, \quad g = 397.$$

(4) $p = 3,\ n = 18$:

$$\begin{aligned} G_4(i,j) \ \cong \ & \langle a, b | a^{18} = b^2 = 1, a^2 = x, x^b = y, [x,y] = x^{3i} y^{-3i}, \\ & y^a = x^{-1} y^{-1}, (ab)^3 = x^{3j} y^{-3j} \rangle, \end{aligned}$$

where $(i,j) = (0,0), (0,1), (1,0), (1,1)$ or $(1,-1)$;

$$\mathcal{M}_4(i,j,l) = \mathcal{M}(G_4(i,j); a^l, b),$$

where $l = 1$ for $(i,j) = (0,0)$ and $l = \pm 1$ for the other cases.
$g = 55$ for $(i,j,l) = (0,0,1)$ and $(1,0,\pm 1)$;
$g = 163$ for $(i,j,l) = (0,1,\pm 1)$, and $(1,\pm 1,\pm 1)$.

(5) $p \geq 5$, $n = p^2 s$, s is a even divisor of $p-1$ and e is of order sp^2 in $\mathbb{Z}^*_{p^3}$:

$$G_5(p,s) \cong \langle a, x | a^{sp^2} = x^{p^3} = 1, a^x = a^e \rangle,$$

$$\mathcal{M}_5(p,s,j) = \mathcal{M}(G_1; a^j, a^{\frac{p^2 s}{2}} c) \quad \text{where} \quad j \in \mathbb{Z}^*_{p^2 s},$$

$$g = 1 + \tfrac{1}{4} p^3 (sp^2 - 4) \text{ for } 4 \mid s; \; g = 1 + \tfrac{1}{4} p^3 (sp^2 - 4) \text{ for } 4 \nmid s.$$

Moreover, the above groups and maps are uniquely determiend by the given parameters.

3.3 $|P| = p^3$

Theorem

Suppose that $|P| = p^3$. Then G and $\mathcal{M}$ are given by

(1) *Define three affine subgroups and the corresponding maps:*

(1.1) $G_{11}(p,n) = T : \langle x \rangle$,

where $x = ||e, d\lambda, f\lambda; f, e + d\varepsilon, f\varepsilon + d\lambda; d, f, e + d\varepsilon||$,

where $p \geq 2$, $n \mid p^3 - 1$ *but* $n \nmid p^2 - 1$; *and* $e + f\beta + d\beta^2$ *is a fixed element of order* n *in* $\mathbb{F}_{p^3}^*$.

$\mathcal{M}_{11}(p,n,i,j) = \mathcal{M}(G_{11}(p,n); x^i, t_{(1,0,0)} x^{\frac{jn}{2}})$,

where $i \in \mathbb{Z}_n^*/(\mathbb{Z}_n^*)^{3+}$, $j = 0$ *for* $p = 2$, *and* $j = 1$ *for* $p \geq 3$.

(1.2) $G_{12}(p, h, d) = T : \langle x \rangle$,

$x = ||1, 1, 0; 1, 0, 0; 0, 0, 1||$ for $p = 2$ and $n = 3$;

$x = ||e, f\theta, 0; f, e, 0; 0, 0, d||$ for $p \geq 3$,

where $(e + f\alpha, d) \in \mathbb{F}^*_{p^2} \times \mathbb{F}^*_p$ such that $(-1, -1) \in \langle (e + f\alpha, d) \rangle$ and $e + f\alpha$ is a fixed element of order h, where $h \mid p^2 - 1$ but $h \nmid p - 1$, and set $n = [h, |d|]$.

$\mathcal{M}_{12}(p, h, d, i, j) = \mathcal{M}(G_{12}(p, h, d); x^i, t_{(1,0,1)} x^{\frac{jn}{2}})$,

where $i \in \mathbb{Z}^*_n / (\mathbb{Z}^*_n)^{2+}$, $j = 0$ for $p = 2$, and $j = 1$ for $p \geq 3$.

(1.3) $G_{13}(p, t_1, t_2, t_3) = T : \langle x \rangle,$

$x = [t_1; t_2; t_3],$

where $p \geq 5$, let $(t_1, t_2, t_3) \in \mathbb{Z}_p^* \times \mathbb{Z}_p^* \times \mathbb{Z}_p^*$ such that $(-1, -1, -1) \in \langle (t_1, t_2, t_3) \rangle$ and t_1, t_2 and t_3 are distinct integer, and set $n = [|t_1|, |t_2|, |t_3|] \geq 4$.

$\mathcal{M}_{13}(p, t_1, t_2, t_3, i) = \mathcal{M}(G_{13}(p, t_1, t_2, t_3); x^i, t_{(1,1,1)} x^{\frac{n}{2}}),$

where $i \in \mathbb{Z}_n^*/(\mathbb{Z}_n^*)^{2+}$ if $t_{i_1}^k = t_{i_1}$, $t_{i_2}^k = t_{i_3}$ and $t_{i_3}^k = t_{i_2}$ for some $k \in \mathbb{Z}_n^*$; $i \in \mathbb{Z}_n^*/(\mathbb{Z}_n^*)^{3+}$ if $t_{i1}^k = t_{i_2}$, $t_{i_2}^k = t_{i_3}$ and $t_{i_3}^k = t_{i_1}$ for some $k \in \mathbb{Z}_n^*$; and $i \in \mathbb{Z}_n^*$ other cases, where $\{i_1, i_2, i_3\} = \{1, 2, 3\}$.

(2) $G_2(p,t_1,t_2) = \langle a,b,x | a^{p^2} = b^p = x^n = 1, [a,b] = 1, a^x = a^{t_1}, b^x = b^{t_2} \rangle$,

where $p \geq 5$, let $(t_1,t_2) \in \mathbb{Z}^*_{p^2} \times \mathbb{Z}^*_p$ such that $|t_1| \mid (p-1)$, $(-1,-1) \in \langle (t_1,t_2) \rangle$ and $t_1 \not\equiv t_2 \pmod{p}$; and set $n = [|t_1|, |t_2|] \geq 4$.

$\mathcal{M}_2(p,t_1,t_2,i) = \mathcal{M}(G_2(p,t_1,t_2); x^i, abx^{\frac{n}{2}})$,

where $i \in \mathbb{Z}^*_n$.

(3) $G_3(p,n) = \langle a, x | a^{p^3} = x^n = 1, a^x = a^t \rangle$,

where $p \geq 3$, n is a even divisor of $p-1$ with $n \geq 2$, and let t be any fixed element of order n in $\mathbb{Z}^*_{p^3}$.

$\mathcal{M}_3(p,n,i) = \mathcal{M}(G_3(p,n); x^i, ax^{\frac{n}{2}})$,

where $i \in \mathbb{Z}^*_n$.

(4) Define two groups:

(4.1) $G_{41}(p, t_1, t_2) = \langle a, b, x | a^p = b^p = c^p = x^n = 1, [a, b] = c, a^x = a^{t_1}, b^x = b^{t_2}, c^x = c^{t_1 t_2} \rangle$,

where $p \geq 5$, let $(t_1, t_2) \in \mathbb{Z}_p^* \times \mathbb{Z}_p^*$ such that $(-1, -1) \in \langle (t_1, t_2) \rangle$ and $t_1 \neq t_2$, and set $n = [|t_1|, |t_2|] \geq 4$.

$\mathcal{M}_{41}(p, t_1, t_2, i) = \mathcal{M}(G_{41}(p, t_1, t_2); x^i, abc^{\frac{p-1}{2}} x^{\frac{n}{2}})$,

where $i \in \mathbb{Z}_n^*/(\mathbb{Z}_n^*)^{2+}$ if $t_1^k = t_2$ and $t_2^k = t_1$ for some $k \in \mathbb{Z}_n^*$; $i \in \mathbb{Z}_n^*$ for other cases.

(4.2) $G_{42}(p,n)=\langle a,b,x | a^p=b^p=c^p=x^n=1,[a,b]=c,a^x=a^{e_1}b^{f_1},b^x=a^{e_2}b^{f_2},c^x=c\rangle$,

$(e_1,f_1,e_2,f_2)=(1,1,1,0)$ for $p=2$ and $n=3$;

$(e_1,f_1,e_2,f_2)=(e,f\theta,f,e)$ for $p\geq 3$,

where $n \mid p^2-1$ but $n\nmid p-1$, $e+f\alpha$ is a fixed element of order n in $\mathbb{F}^*_{p^2}$.

$$\mathcal{M}_{42}(p,n,i,j)=\mathcal{M}(G_{42}(p,n);x^i,ac^{\frac{jef\theta(1-e+f)}{4(e-1)}}x^{\frac{jn}{2}}),$$

where $i=\pm 1$ and $j=0$ for $p=2$, or $i\in\mathbb{Z}_n^*\cap\{1,2,\cdots,\frac{n}{2}\}$ and $j=1$ for $p\geq 3$.

Theorem

Suppose that $|P| = p^4$. Then G and $\mathcal{M}$ are given by

(1) $G_1(p,h) = \langle a, b, x | a^{p^3} = b^p = x^h = 1, a^b = a^{1+p^2}, a^x = a^e, b^x = b \rangle$,

where $p \geq 3$, $n = ph$ and h any even divisor of $p-1$, and let e be any fixed element of order h in $\mathbb{Z}^*_{p^2}$.

$\mathcal{M}_1(p,h,i,j) = \mathcal{M}(G_1(p,h); b^i x^j, a x^{\frac{h}{2}})$,

where $i \in \mathbb{Z}^*_p$ and $j \in \mathbb{Z}^*_h$.

(2) $G_2(p,h) = \langle a, b, x | a^{p^2} = b^p = c^p = x^h = 1, [a,b] = c, [c,a] = [b,c] = 1, a^x = a^e, b^x = b \rangle.$

where $p \geq 3$, $n = ph$ and $h \geq 2$ is an even divisor $p-1$, and let e be any fixed element of order h in $\mathbb{Z}_{p^2}^*$.

$\mathcal{M}_2(p,h,i) = \mathcal{M}(G_2(p,e); bx^i, acx^{\frac{h}{2}}),$

where $i \in \mathbb{Z}_h^*$.

(3) $G_3(p,t_1,t_2)=\langle a,b,c,x | a^{p^2}=b^p=c^p=x^h=1, a^b=a^{1+p}, [a,c]=[b,c]=1, a^x=a^{t_1}, b^x=b, c^x=c^{t_2}\rangle$,

where $p\geq 5$, $n=ph$ and let $h \mid p-1$, let $(t_1,t_2)\in \mathbb{Z}_{p^2}^*\times\mathbb{Z}_p^*$ such that $|t_1|=h$, $t_1\neq t_2$ and $\langle (t_1,t_2)\rangle$ contains $(-1,-1)$.

$\mathcal{M}_3(p,t_1,t_2,i,j)=\mathcal{M}(G_3(p,t_1,t_2); b^ix^j, acx^{\frac{h}{2}})$,

where $i\in\mathbb{Z}_p^*$ and $j\in\mathbb{Z}_h^*$.

(4) $G_4(p, t_1, t_2) = \langle a, b, d, x | a^p = b^p = c^p = d^p = x^h = 1, [a, b] = c, [a, c] = [b, c] = [a, d] = [b, d] = 1, a^x = a, b^x = b^{t_1}, d^x = d^{t_2} \rangle$,

where $p \geq 5$, $n = ph$, let $(t_1, t_2) \in \mathbb{Z}_p^* \times \mathbb{Z}_p^*$ such that $t_1 = \theta^{\frac{p-1}{h}}$, $t_1 \neq t_2$ and $\langle (t_1, t_2) \rangle$ contains $(-1, -1)$, let $h = [|t_1|, |t_2|]$ with $h \geq 4$ is even.

$\mathcal{M}_4(p, t_1, t_2, i) = \mathcal{M}(G_4(p, t_1, t_2); ax^i, bdx^{\frac{h}{2}})$,

where $i \in \mathbb{Z}_h^*$.

(5) $G_5(p,h) = \langle a, b, x | a^{p^2} = b^p = c^p = x^h = 1, [a,b] = c, [a,c] = 1, [c,b] = a^{ip}, a^x = a^t, b^x = b \rangle$,

where $p \geq 3$ and let t be any fixed element of order h in $\mathbb{Z}_{p^2}^*$.

$\mathcal{M}_5(p,h,j,k) = \mathcal{M}(G_5(p,h); b^j x^k, a x^{\frac{h}{2}})$,

where $j \in \mathbb{Z}_p^* \cap \{1, 2, \cdots, \frac{p-1}{2}\}$ and $k \in \mathbb{Z}_h^*$.

(6) $G_6(p, t_1, t_2, t_3) = \langle a, b, x | a^{p^2} = b^p = c^p = x^h = 1, [a, b] = c, [c, a] = a^p, [c, b] = 1, a^x = a^{t_1} c^{t_3}, b^x = b^{t_2} c^{\frac{1-t_2}{2}} \rangle$,

$t_1 \not\equiv t_2 (\mathrm{mod}\ p)$, $(-1, -1) \in \langle (t_1, t_2) \rangle$ and
$t_1^h - \frac{pht_3}{2} \equiv 1 (\mathrm{mod}\ p^2)$.

$\mathcal{M}_6(p, t_1, t_2, t_3, i, j, k) =$
$\mathcal{M}(G_6(p, t_1, t_2, t_3); c^i x^j, ab^k c^{-k - \frac{t_3}{1-t_1}} x^{\frac{h}{2}})$,

where $i, k \in \mathbb{Z}_p^*$ and $j \in \mathbb{Z}_h^*$.

(7) Define three affine subgroups and the corresponding maps:

(7.1) $G_{71}(p,t) = \langle a, b, x \mid a^p = b^p = c^p = d^p = x^h = 1, [a,b] = c, [c,a] = 1, [c,b] = d, a^x = a^t, b^x = b \rangle$,

where $p \geq 5$ and let t be any fixed element of order h in $\mathbb{Z}_p^*$;

$\mathcal{M}(p,t,i) = \mathcal{M}(G_{71}(p,t); bx^i, ax^{\frac{h}{2}})$,

where $i \in \mathbb{Z}_p^*$.

(7.2) $G_{72}(p,t) = \langle a, b, x \mid a^p = b^p = c^p = d^p = x^h = 1, [a,b] = c, [c,a] = 1, [c,b] = d, a^x = a, b^x = b^t \rangle$,

where $p \geq 5$ and let t be any fixed element of order h in $\mathbb{Z}_p^*$;

$\mathcal{M}_{72}(p,t,i) = \mathcal{M}(G_{72}(p,t); ax^i, bx^{\frac{h}{2}})$,

where $i \in \mathbb{Z}_h^*$.

(7.3) $G_{73}(p, t_1, t_2) = \langle a, b, x \mid a^p = b^p = c^p = d^p = x^h = 1, [a, b] = c, [c, a] = 1, [c, b] = d, a^x = a^{t_1} c^{\frac{t_1 - 1}{2}}, b^x = b^{t_2} \rangle,$

where $p \geq 5$ and let $(t_1, t_2) \in \mathbb{Z}_p^* \times \mathbb{Z}_p^*$ such that $t_1 t_2 \equiv 1 (\mathrm{mod}\ p)$, $t_1 \neq t_2$ and $(-1, -1) \in \langle (t_1, t_2) \rangle$;

$\mathcal{M}_{73}(p, t_1, t_2, i, j) = \mathcal{M}(G_{73}(p, t_1, t_2), c^i x^j, abx^{\frac{h}{2}}),$

where $i \in \mathbb{Z}_p^*$ and $j \in \mathbb{Z}_h^*$.

(8) Define two subgroups and the corresponding maps:

(8.1) $G_{81}(2,2) = \langle a, b \mid a^8 = b^2 = 1, a^b = a^{-1} \rangle$

$\mathcal{M}_{81}(2,2) = \mathcal{M}(G_{81}(2,2), b, ab).$

(8.2) $G_{82}(2,2) = \langle a, b, c \mid a^4 = b^2 = c^2 = [a,c] = [b,c] = 1, a^b = a^{-1} \rangle$

$\mathcal{M}_{82}(2,2,i) = \mathcal{M}(G_{82}(2,2), b, ab).$

4. Further woks:

1. NORM of order p^3
2. Classify RM of order p^3 with multiple edges.

Thank You Very Much !