2-Arc-Transitive Metacyclic Covers of Complete Graphs

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Rogla, Slovenian, July 2, 2014

Outline

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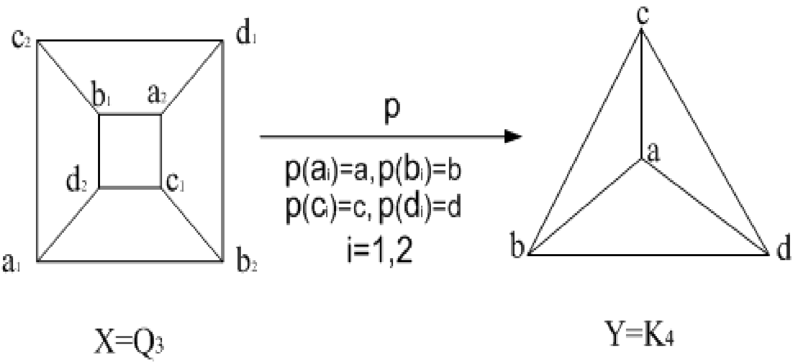
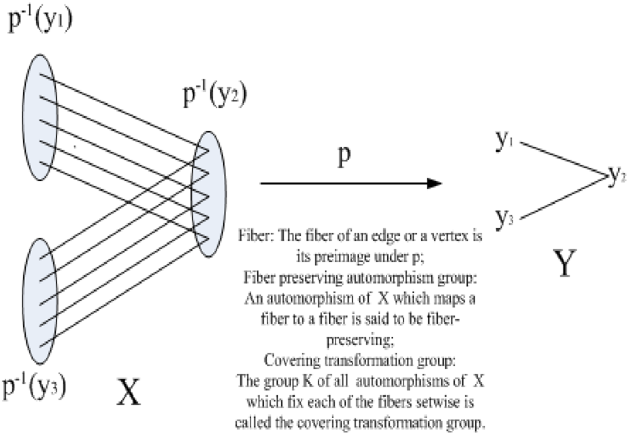
1. Introduction

Definitions

- Base graph and Covering graph:
  A graph X is called a covering of a graph Y with the projection $p : X \to Y$ if there is a surjection $p : V(X) \to V(Y)$ such that $p|_{N(x)} : N(x) \to N(y)$ is a bijection for any $y \in V(Y)$ and $x \in p^{-1}(y)$.

  X: *Covering graph*; Y: *base graph*; A covering p is *n-fold* if $|p^{-1}(y)| = n$ for each $y \in V(Y)$.


c2
d1
b1
a2
d2
c1
a1
b2
p
p(ai)=a, p(bi)=b
p(ci)=c, p(di)=d
i=1,2
c
a
b
d
X=Q3
Y=K4


$p^{-1}(y_1)$
$p^{-1}(y_2)$
$p^{-1}(y_3)$
X
p
y_1
y_2
y_3
Y
Fiber: The fiber of an edge or a vertex is its preimage under p;
Fiber preserving automorphism group:
An automorphism of X which maps a fiber to a fiber is said to be fiber-preserving;
Covering transformation group:
The group K of all automorphisms of X which fix each of the fibers setwise is called the covering transformation group.

- Fiber:
  The *fiber* of an edge or a vertex is its preimage under p.
- Fiber preserving automorphism group:
  An automorphism of X which maps a fiber to a fiber is said to be *fiber-preserving*.
- Covering transformation group:
  The group K of all automorphisms of X which fix each of the fibers setwise is called the *covering transformation group*.

  It is easy to see that if X is connected then the action of K on the fibers of X is necessarily semiregular; that is, $K_v = 1$ for each $v \in V(X)$. In particular, if this action is regular we say that X is a *regular cover* of Y.

Lifting: $\alpha \in \mathrm{Aut}(Y)$ *lifts* to an automorphism $\overline{a} \in \mathrm{Aut}(X)$ if $\alpha p = p\overline{a}$.

Question: Given a graph Y, a group K and $H \leq \mathrm{Aut}(Y)$, find all the connected regular coverings $Y \times_f K$ on which H lifts.

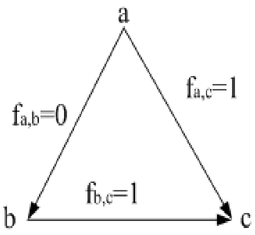
Combinatorial description of a covering

Voltage assignment f: graph Y, finite group K
a function $f : A(Y) \to K$ s. t. $f_{u,v} = {f_{v,u}}^{-1}$ for each $(u, v) \in A(Y)$.

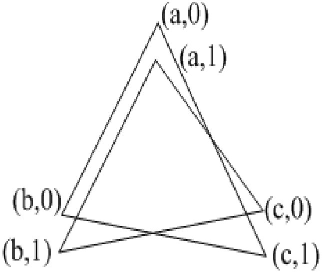
Voltage graph $Y \times_f K$: vertex set $V(Y) \times K$,
arc-set $\{((u, g), (v, g f_{u,v}) \mid (u, v) \in A(Y), \ g \in K\}$.

Remark:
Voltage graph $Y \times_f K$ is a covering of Y;

$$f: A(Y) \longrightarrow K=\{1,0\}$$



$Y=K_3$



$Y \times_f K$

Classification of 2-arc-transitive Graphs

Praeger's Reduction Theorem

Every finite connected 2-arc-transitive graphs X is:

(1) *Quasiprimitive Type:* every non-trivial normal subgroup of $\mathrm{Aut}X$ acts transitively on $V(X)$,

(2) *Bipartite Type:* every non-trivial normal subgroup of $\mathrm{Aut}X$ has at most two orbits on $V(X)$ and at least one of normal subgroups of $\mathrm{Aut}X$ has exactly two orbits on $V(X)$.

(3) *Covering Type:* covers of graphs in (1) and (2).

- C.E. Praeger, On a reduction theorem for finite, bipartite, 2-arc-transitive graphs, *Australas J. Combin.* **7**(1993), 21-36.

For the quasiprimitive type and bipartite type, a lot of results have appeared:

- A.A. Ivanov and C.E. Praeger, On finite affine 2-arc-transitive graphs, *Europ. J. Combin.* **14** (1993), 421–444.
- C.E. Praeger, An O'Nan-Scott theorem for finite quasiprimitive permutation groups and an application to 2-arc transitive graphs, *J. London Math. Soc.* **47**(1993), 227-239.
- C.E. Praeger, Finite quasiprimitive graphs, in: *Surveys in Combinatorics, London Mathematical Society Lecture Note Series,* **260**, Cambridge University Press, Cambridge, 1997, pp. 65–85.
- C.E. Praeger, Bipartite 2-arc-transitive graphs, *Australas J. Combin.* **7**(1993), 21–36.
- R. Baddeley, Two-arc transitive graphs and twisted wreath products, *J.Algebr.Comb.* **2**(1993), 215–237.

- C.H. Li, On finite s-transitive graphs of odd order, *J. Comb. Theory B* **81**(2001), 307–317.
- C.H. Li, Z.P. Lu, D. Marušič, On Primitive Permutation groups with small suborbits and their orbital graphs, *J. Algebra* **279**(2004), 749–770.
- X.G. Fang, G. Havas and C.E. Praeger, On the automorphism groups of quasiprimitive almost simple graphs, *J. Algebra* **222**(1999), 271–283.
- X.G. Fang, C.H. Li and C.E. Praeger, The locally 2-arc transitive graphs admitting a Ree simple group, *J. Algebra* **282**(2004), 638–666.

The results concerning the 2-arc-transitive regular covers of complete graphs

- S.F. Du, D. Marušič and A.O. Waller, On 2-arc-transitive covers of complete graphs, *J. Comb. Theory, Ser. B*, **74**(1998), 276–290. (for the covering transformation group is cyclic or $\mathbb{Z}_p^2$)
- S.F. Du, J.H. Kwak and M.Y. Xu, On 2-arc-transitive covers of complete graphs with covering transformation group $\mathbb{Z}_p^3$, *J. Combin. Theory, B* **93** (2005), 73–93.

2. Metacyclic covers of complete graph

Any metacyclic group can be presented by

$$K = \langle a, b \mid a^d = 1, b^m = a^t, a^b = a^r \rangle$$

where $r^m \equiv 1 (\mathrm{mod}\ d), t(r-1) \equiv 0 (\mathrm{mod}\ d)$.

If d is even, $m = 2$, $r = -1$ and $t = d/2$, then $K \cong Q_{2d}$, so called the generalized quaternion group of order $2d$;

If $m = 2$, $r = -1$ and $t = 0$, then $K \cong D_{2d}$, the dihedral group of order $2d$.

Note that $Q_4 \cong \mathbb{Z}_4$ and $D_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Theorem

Let X be a connected regular cover of the complete graph K_n $(n \geq 4)$ whose covering transformation group K is nontrivial metacyclic and whose fibre-preserving automorphism group acts 2-arc-transitively on X. Then X is isomorphic to one of covers below:

(1) *The canonical double cover $K_{n,n} - nK_2$ with $K \cong \mathbb{Z}_2$;*

(2) $n = 4$, $AT_D(4, 6)$ *with* $K \cong D_6$;

(3) $n = 4$, $AT_Q(4, 12)$ *with* $K \cong Q_{12}$;

(4) $n = 5$, $AT_D(5, 6)$ *with* $K \cong D_6$;

(5) $n = 1 + q \geq 4$, $AT_Q(1 + q, 2d)$ *with* $K \cong Q_{2d}$, *where* $d \mid q - 1$ *and* $d \nmid \frac{1}{2}(q - 1)$;

(6) $n = 1 + q \geq 6$, $AT_D(1 + q, 2d)$ *with* $K \cong D_{2d}$, *where* $d \mid \frac{1}{2}(q - 1)$ *and* $d \geq 2$.

For the case $n = 4$ the following are the two covers of K_4 with respective covering transformation group $K = \langle a, b\rangle \cong D_6$ and Q_{12}, where $V(K_4) = \{1, 2, 3, 4\}$:

(1) $AT_D(4, 6) = K_4 \times_f D_6$, with the voltage assignment $f : A(K_4) \to D_6$ defined by

$$f_{1,2} = b, f_{1,3} = ba, f_{1,4} = ba^{-1}, f_{2,3} = ba^{-1}, f_{2,4} = ba, f_{3,4} = b;$$

(2) $AT_Q(4, 12) = K_4 \times_f Q_{12}$, with the voltage assignment $f : A(K_4) \to Q_{12}$ defined by

$$f_{1,2} = b, f_{1,3} = ba^2, f_{1,4} = ba^4, f_{2,3} = b, f_{2,4} = ba^3, f_{3,4} = b.$$

For the case $n = 5$ this is one cover of K_5 with the covering transformation group $K = \langle a, b\rangle \cong D_6$, where $V(K_5) = \{1, 2, 3, 4, 5\}$:

(3) $AT_D(5, 6) = K_5 \times_f D_6$, with the voltage assignment $f : A(K_5) \to D_6$ defined by

$$f_{1,2} = ab, f_{1,3} = b, f_{1,4} = ba, f_{1,5} = b, f_{2,3} = ba,$$
$$f_{2,4} = b, f_{2,5} = b, f_{3,4} = ab, f_{3,5} = b, f_{4,5} = b.$$

Next, let $\mathrm{GF}(q)$ be the field of order q where q is odd, and let $\mathrm{GF}(q)^* = \langle\theta\rangle$. We identify $V(K_{1+q})$ with $\mathrm{PG}(1,q) = \mathrm{GF}(q) \cup \{\infty\}$. Then the following two families of 2-arc-transitive covers of K_{1+q} with the respective covering transformation groups $K = \langle a, b\rangle \cong Q_{2d}$ and D_{2d}:

(4) $AT_Q(1+q, 2d) = K_{1+q} \times_f Q_{2d}$, where $d \mid q-1$ and $d \nmid \frac{1}{2}(q-1)$;

(5) $AT_D(1+q, 2d) = K_{1+q} \times_f D_{2d}$, where $d \mid \frac{1}{2}(q-1)$ and $d \geq 2$,

and for both covers, the voltage assignments $f : A(K_{1+q}) \to K$ are given by:

$$f_{\infty,i} = b;\;\; f_{i,j} = ba^h \;\text{ if }\; j - i = \theta^h \;\text{ for }\; i, j \neq \infty.$$

For the case K is cyclic or is isomorphic to $\mathbb{Z}_p^2$, we have the following remark:

- S.F. Du, D. Marušič and A.O. Waller, On 2-arc-transitive covers of complete graphs, *J. Comb. Theory, B* **74**(1998), 276–290.

Remark

Suppose that X is a connected regular cover of the complete graph K_n $(n \geq 4)$ whose covering transformation group K is either nontrivial cyclic or $\mathbb{Z}_p^2$ and whose fibre-preserving automorphism group acts 2-arc-transitively on X. Then X is isomorphic to one of $K_{n,n} - nK_2$ with $K \cong \mathbb{Z}_2$; $AT_Q(1+q,4)$ with $K \cong \mathbb{Z}_4$ and $q \equiv 3 (\mathrm{mod}\ 4)$; and $AT_D(1+q,4)$ with $K \cong \mathbb{Z}_2^2$ and $q \equiv 1 (\mathrm{mod}\ 4)$. Moreover, $\mathrm{Aut}(AT_i(1+q,4))/K \cong \mathrm{P\Gamma L}(2,q)$, where $i \in \{Q, D\}$.

3. Outline of proof

Base graph $Y = K_n$,

covering graph X,

covering transformation group K is a metacyclic group:

$$K = \langle a, b \mid a^d = 1, b^m = a^t, b^{-1}ab = a^r \rangle,$$

where $t(r-1) \equiv 0 (\mathrm{mod}\ d)$, $r^m \equiv 1 (\mathrm{mod}\ d)$

$\overline{A}$=2-arc-transitive subgroup of $\mathrm{Aut}(Y)$ which will be lifted,

$\overline{A}$ is 3-transitive on $V(Y)$,

$\overline{A}$ should satisfy one of the following cases:

(1) $\overline{A} = S_4$;

(2) $\overline{A} = \mathbb{Z}_2^m \rtimes \mathrm{GL}(m, 2)$ or $\overline{A} = \mathbb{Z}_2^4 \rtimes A_7$;

(3) $\overline{A}$ is an almost simple group, and the socle of $\overline{A}$ is either 3-transitive, or $\mathrm{PSL}(2, q)$.

A=the fiber preserving subgroup of $\mathrm{Aut}(X)$,

$A/K = \overline{A}$,

$\Longrightarrow$ the problem of group extension.

K is abelian

Lemma

Suppose that the covering transformation group K is abelian metacyclic. Then K is isomorphic to $\mathbb{Z}_2$, $\mathbb{Z}_4$, or $\mathbb{Z}_{s\cdot 2^\ell} \times \mathbb{Z}_{2^\ell}$, where $\ell \geq 1$ and $s \in \{1, 2, 4\}$. In particular, K is a 2-group.

Lemma

For any positive integers t_1 and t_2, $\mathrm{Aut}(\mathbb{Z}_{t_1} \times \mathbb{Z}_{t_2})$ does not contain a nonabelian simple subgroup.

Key lemma

If the covering transformation group K is abelian metacyclic, then the covering graph X is isomorphic to one of $K_{n,n} - nK_2$ with $K \cong \mathbb{Z}_2$, $AT_Q(1+q,4)$ with $K \cong \mathbb{Z}_4$, and $AT_D(1+q,4)$ with $K \cong \mathbb{Z}_2^2$.

Proof: Set $K = \langle a\rangle \times \langle b\rangle$, where $|a| = s2^\ell$, $|b| = 2^\ell$ and $s \in \{1,2,4\}$,

and if $\ell = 1$ then $s \neq 1$.

(1) Assume $\overline{A} = S_4$ with the degree $n = 4$.

Let $K_1 = \langle a^2, b^2\rangle$. Then K_1 *char* K and $K/K_1 \cong \mathbb{Z}_2^2$.

By the group K_1 the projection $X \to K_n$ is factorized as

$X \to Y \to K_n$, where Y is a cover of K_n with the covering transformation group $\mathbb{Z}_2^2$.

By remark, we know that if $K/K_1 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$,

then $Y \cong AT_D(1+q,4)$ and $n = q+1$, where $q \equiv 1 (\mathrm{mod}\ 4)$.

(2) Let $\overline{A} = \mathbb{Z}_2^m \rtimes \mathrm{GL}(m,2)$ with $m \geq 3$ or $\overline{A} = \mathbb{Z}_2^4 \rtimes A_7$.($\mathrm{Aut}K$ contains a nonabelian simple subgroup, which is impossible)

(3) Suppose that $\overline{A}$ is an almost simple group.(K is cyclic or is isomorphic $\mathbb{Z}_2^2$, which contradicts our hypothesis too.)

K is nonabelian

Key lemma

If K is nonabelian, then it is one of the following two cases:

(1) *K contains a cyclic subgroup N of index 2 such that $N \lhd A$;*

(2) *$K = \langle a, b \mid a^d = b^4 = 1, a^b = a^r \rangle$, where d is odd, $r^4 \equiv 1 (\mathrm{mod}\ d)$, $r^2 \not\equiv 1 (\mathrm{mod}\ d)$ and $(d, r-1) = 1$.*

Case 1: K contains a cyclic subgroup N of index 2 such that $N \lhd A$;

Lemma

Suppose that there exists a cyclic subgroup N of K of index 2 such that $N \lhd A$. Then X is the cyclic regular cover of $K_{n,n} - nK_2$ with the covering transformation group N, whose fibre (N-orbits) preserving automorphism group acts 2-arc-transitively.

Proposition

Let X be a connected regular cover of $K_{n,n} - nK_2$ $(n \geq 4)$ with a non-trivial cyclic covering transformation group $\mathbb{Z}_d$ whose fiber-preserving automorphism group acts 2-arc-transitively. Then one of the following holds:

(1) $n = 4$ *and* X *is isomorphic to the unique* $\mathbb{Z}_d$*–cover, where* $d = 2, 3, 6$*;*

(2) $n = 5$ *and* X *is isomorphic to the unique* $\mathbb{Z}_3$*–cover;*

(3) $n = q + 1 \geq 5$ *and* $X \cong K^{2d}_{1+q}$*, defined just below.*

Graphs K_{1+q}^{2d}: Let $q = r^l$ for an odd prime r

and $\mathrm{GF}(q)^* = \langle\theta\rangle$ the multiple group of the field $\mathrm{GF}(q)$ of order q.

$V(K_{q+1,q+1} - (q+1)K_2) = \{i, i' | i \in \mathrm{PG}(1,q)\}$,

the missing matching consists of all pairs $[i, i']$.

Define a voltage graph $K_{q+1}^{2d} = (K_{1+q,1+q} - (1+q)K_2) \times_f \mathbb{Z}_d$, where

$f_{\infty',i} = f_{\infty,j'} = \overline{0}\ \text{ for } i, j \neq \infty;\ f_{i,j'} = \overline{h}\ \text{ if } j - i = \theta^h,\ \text{ for } i, j \neq \infty.$

Key lemma

Suppose that $n = 4$. *Then* X *is isomorphic to* $AT_D(4,6)$ *or* $AT_Q(4,12)$.

Proof:

Since there exists a unique $\mathbb{Z}_d$-cover of $K_{4,4} - 4K_2$ satisfying our condition with $d = 3$ or 6, it suffices to define a $2d$-fold cover of K_4 directly, which also satisfies our condition and is a $\mathbb{Z}_d$-cover of $K_{4,4} - 4K_2$.

Step 1

We give the structure of A directly.

Step 2

Determination of point stabilizers $H := A_{\tilde{u}} \cong \overline{A}_u$

Step 3

Determination of coset graphs $X(A, H; D)$

(i) *Undirected property* : $D^{-1} = D$

(ii) *The Length of the suborbit is n-1*

(iii) *Connected property* : $A = \langle D \rangle$

Step 4

Show that the coset graph is isomorphic to a voltage graph

Key lemma

Suppose that $n = 5$. Then X is isomorphic to $AT_D(5, 6)$.

Key lemma

Suppose that $n \geqslant 5$. Then X is isomorphic to $AT_Q(1 + q, 2d)$ or $AT_D(1 + q, 2d)$, where $d \geq 3$.

Case 2: $K = \langle a, b \mid a^d = b^4 = 1, a^b = a^r \rangle$, where d is odd, $r^4 \equiv 1 (\mathrm{mod}\ d)$, $r^2 \not\equiv 1 (\mathrm{mod}\ d)$ and $(d, r-1) = 1$.

Proof:

Let T be a lift of $\mathrm{PSL}(2, q)$, that is, $T/K \cong \mathrm{PSL}(2, q)$.

On the one hand, by the structure of K, we get

$$T/K' = (C_T(K)K'/K') \times (K/K') \cong \mathrm{PSL}(2, q) \times \mathbb{Z}_4. \quad (1)$$

On the other hand, let Z be the quotient graph of X induced by K'.

In particular, $(T/K')/(K/K') \cong \mathrm{PSL}(2,q)$ lifts. (Note that in this case $K/K' \cong \mathbb{Z}_4$)

All such covers have been determined: these are $AT_Q(1+q,4)$, where $q \equiv 3 (\mathrm{mod}\ 4)$.

In particular, $\mathrm{PSL}(2,q)$ is lifted to be the following group

$$T/K' \cong SL(2,q)\mathbb{Z}_4. \quad (2)$$

The contradiction between Eq(1) and Eq(2) shows that case (2) is impossible.

Thank You Very Much !