Classification of reflexible edge-transitive embeddings of $K_{m,n}$ and corresponding groups

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SYGN IV, Rogla
(Joint work with Jin Ho Kwak)

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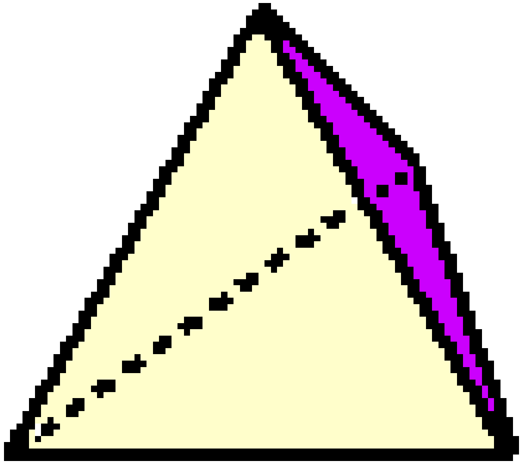
4. Groups represented by product of two cyclic groups

Introductions to maps



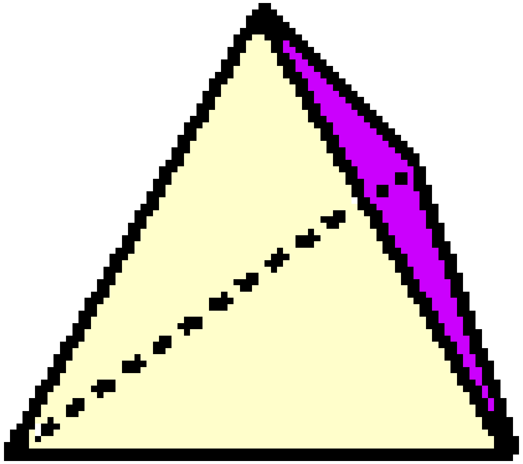
Underlying graph: K_4

Supporting surface: sphere(S_0)

$K_4 \rightarrow S_0$ (2-cell embedding)

A topological map: a $2-cell$ embedding of a graph into a surface.

Introductions to maps



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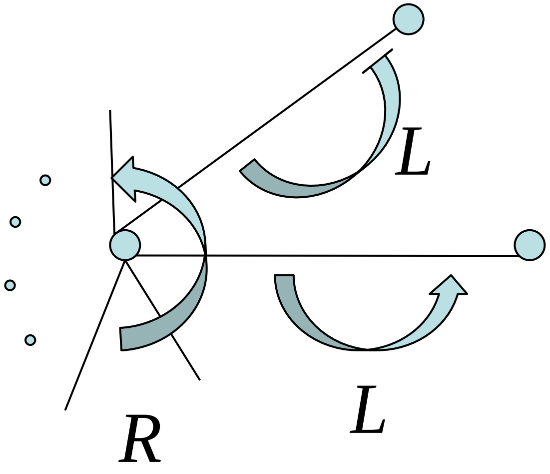
Several descriptions

1. Drawing

2. Combinatorial map: $(D; R, L)$



$D \leftrightarrow$ arcs(incident vertex-edge pairs) set

$R:$ rotation $\quad L:$ arc-reversing involution

$\langle R, L \rangle$ acts transitively on D.

$L^2 = 1$

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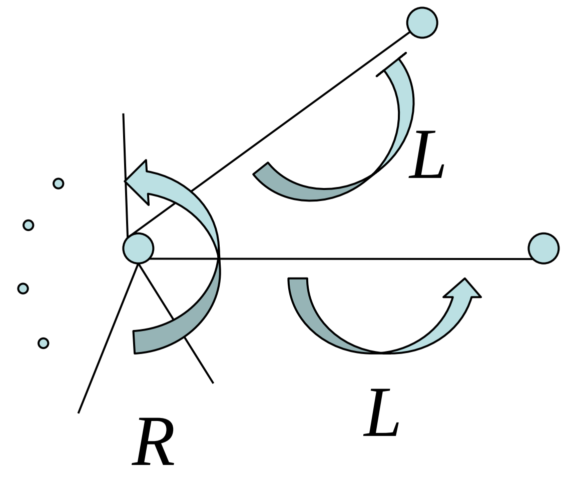

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3. Map subgroup

$(D : R, L)$ with $o(LR^{-1}) = k,\ o(R) = m$ (type (k,m)-map) $\rightarrow$

$T^o(k,m,2) = \left\langle r, \ell \mid r^m = \ell^2 = (\ell r^{-1})^k = 1 \right\rangle$ acts on D by $x^r = x^R, x^\ell = x^L$

$M = Stab(x) \le T^o(k,m,2)$ for some $x \in D$: map subgroup

M: torsion-free subgroup of index $|D|$.

4. Belyi pair: (X, f)

(X, f) is a Belyi pair if

1. X is a Riemann surface of genus g for some $g \geq 1$.

2. $f: X \to \bar{C}$ is a meromorphic fuction with at most three critical valus 0, 1 and ∞. ($\bar{C} = C \cup \{\infty\}$ is the complex Riemann sphere)

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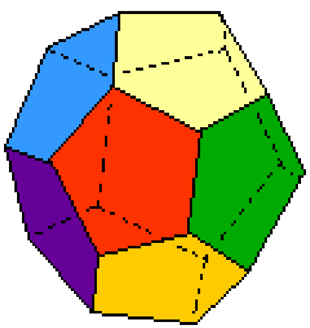

$$f_{\text{dodeca}}(z) = 1728 \frac{\left(z^{10} - 11 z^5 - 1\right)^5 z^5}{\left(z^{20} + 228 z^{15} + 494 z^{10} - 228 z^5 + 1\right)^3}$$

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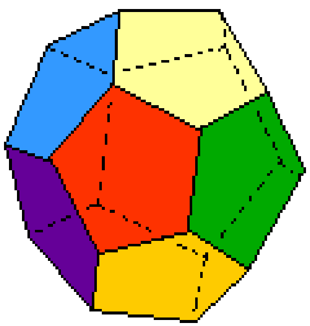

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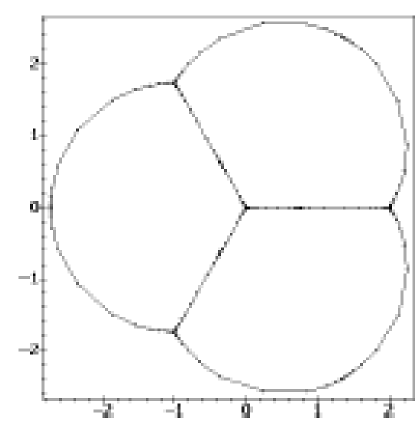
$$f_{tetra}(z) = -64 \frac{(z^3 + 1)^3}{(z^3 - 8)^3 z^3},$$



Introductions to symmetric maps

map automorphisms

1. For an orientable map $\mathfrak{M} = G \to S,$ a (orientation preserving) *map automorphism* is a graph automorphism of G which can be extended to a (oientation preserving) self-homeomorphism of the surface S in the embedding.

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$$* \ |\mathrm{Aut}^{+}(\mathfrak{M})| \ \le \ |\mathrm{D}| = 2|E| \ \le \ |\langle \mathrm{R,L} \rangle|$$

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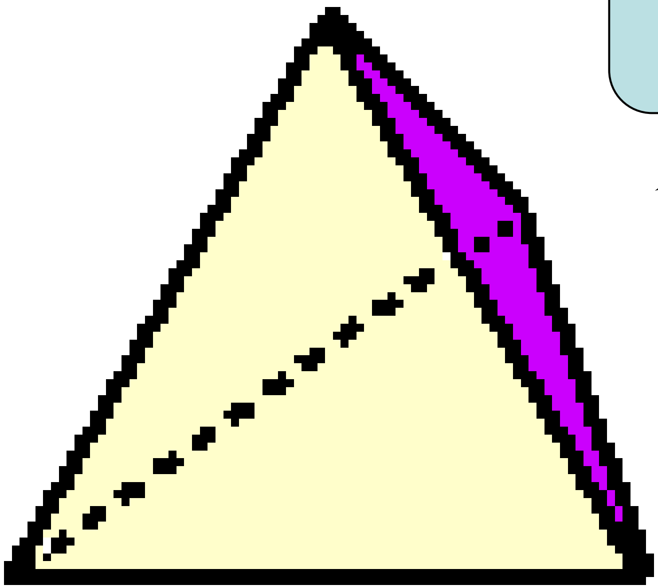
$$* \ |\mathrm{Aut}^+(\mathfrak{M})| \le |\mathrm{D}| = 2|E| \le |\langle \mathrm{R,L} \rangle|$$

One equality holds $\Leftrightarrow$ both equalities hold $\Leftrightarrow$ $\mathrm{Aut}^+(\mathfrak{M}) \simeq \langle R, L\rangle$

In this case, we call $\mathfrak{M}$ an *orientably regular map* or *orientably regular embedding* of G.

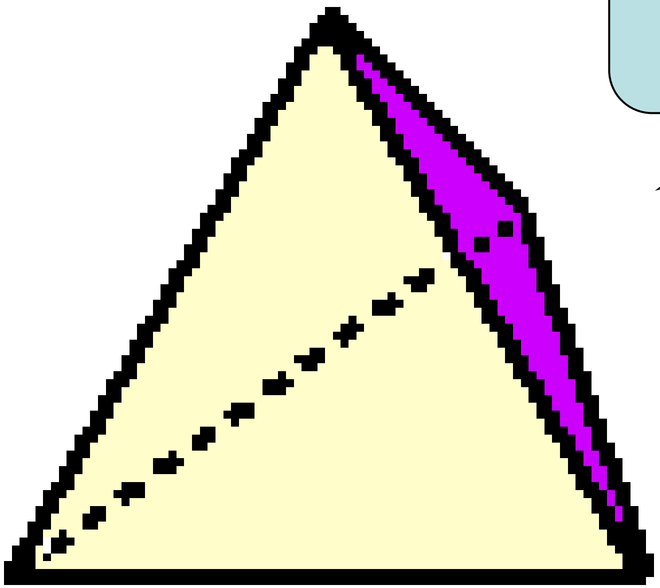


The set of orientation preserving automorphism of tetrahedron: $\approx A_4$

$| A_4 | = 2|E|=12$

The set of all (orientation preserving and orientation reversing) automorphism: $\approx S_4$

$| S_4 | = 4|E|=24$



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Classification of (orientably) regular maps are pursued by three different directions: fixed surface,

fixed automorphism group

fixed graph

Some classification of (orientably) regular map

Complete graph K_n

Orientable: James and Jones(1984), nonorientable:Wilson(1989).

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Complete multipartite graphs $K_{n,\ldots,n}=K_{n*d}$

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$\exists$ an orientably regular embeddings of G $\Rightarrow$

G is symmetric(arc-transitive).

Q What is the most symmetric embeddings of $K_{m,n}$ with $m \neq n$?

Reflexible edge transitive embeddings of Km,n

Orientably regular embeddings of $\mathrm{K}_{n,n} \leftrightarrow$

$n-isobicyclic\ triples$. (G. Jones, R. Nedela, M. Skoviera)

(G, x, y) : n-$isobicyclic$ if

(i) $G = \langle x \rangle \langle y \rangle$ (ii) $\langle x \rangle \simeq \langle y \rangle \simeq \mathbb{Z}_n$ and $\langle x \rangle \cap \langle y \rangle = \{id\}$

(iii) $\exists\ \alpha \in Aut(G)$ interchanging x and y.

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$\leftrightarrow\ r_\sigma, l \in Aut(K_{n,n})$ of the following form:

$r_\sigma = \sigma(0', 1', 2', \ldots.., n\text{-}1'), l = (0,0')(1,1')(2,2')\cdots(n-1, n-1')$,

such that $\sigma(0)$=0 and $|\langle r_\sigma, l \rangle| = 2n^2$.

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reflexible $\leftrightarrow$ (*iv*) $\exists\ \phi \in$ Aut(G) s.t. $x^\phi = x^{-1}, y^\phi = y^{-1} \leftrightarrow \sigma^{-1}(-k) = -\sigma(k)$

An n-isobicyclic triple (G, x, y) $\rightarrow$ an orientably regular embedding of $K_{n,n}$:

(1) Vertex set : $\{g\langle x\rangle \mid g \in G\} \cup \{g\langle y\rangle \mid g \in G\}$ (as partite set).

(2) Edge set := G.

(3) The incidence is given by inclusion.

(4) Local rotation at each vertex.

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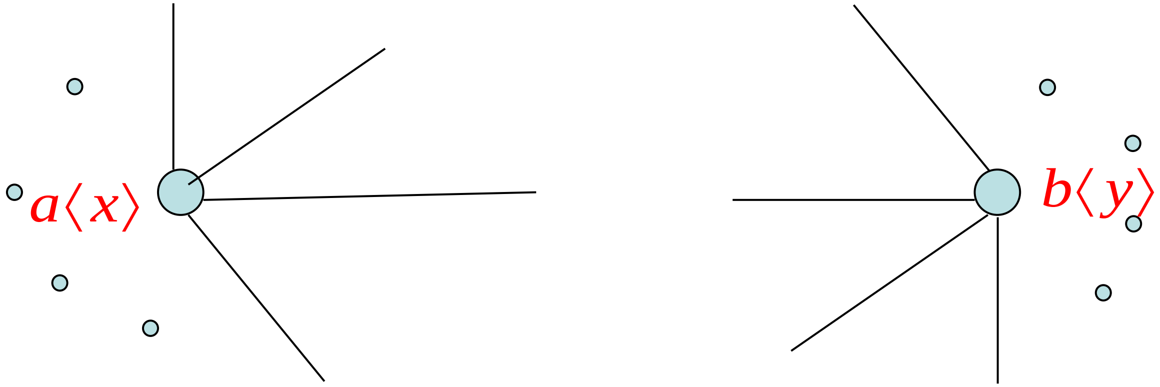

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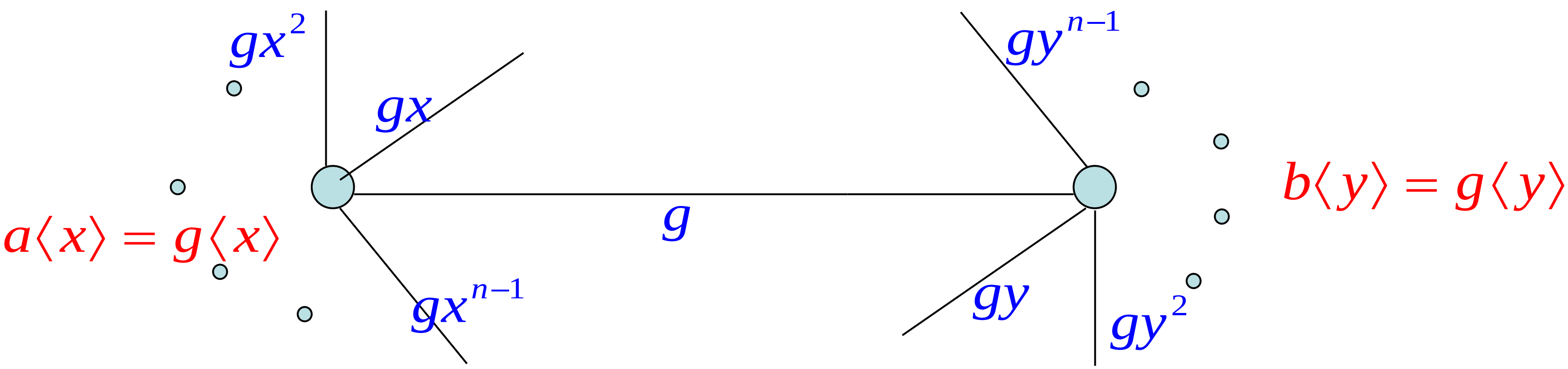

Embedding is defined by $(i),(ii)$ and corresponding embedding is edge transitive.

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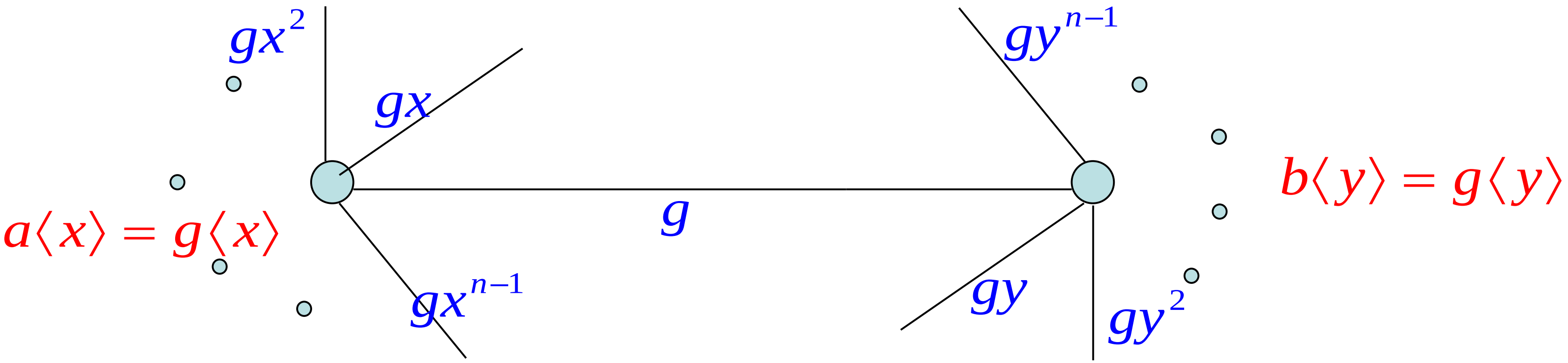

Embedding is defined by $(i), (ii)$ and corresponding embedding is edge transitive.

(iii) $\Rightarrow$ orientably regular.

(iv) $\Rightarrow$ reflexible.

Edge transitive embeddings of $\mathrm{K}_{m,n} \leftrightarrow (m,n)-isobicyclic\ triples$

$(G,x,y):(m,n)$-*isobicyclic* if

(i) $G=\langle x\rangle\langle y\rangle$ (ii) $\langle x\rangle \simeq \mathbb{Z}_n, \langle y\rangle \simeq \mathbb{Z}_m$ and $\langle x\rangle \cap \langle y\rangle = \{id\}$

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$\leftrightarrow\ x_\alpha, y_\beta \in Aut(K_{m,n})$ of the following form:

$x_\alpha = \alpha(0',\ 1',\ 2',\ \ldots\ldots,\ n\text{-}1'),\ \ y_\beta = \beta(0,\ 1,\ 2,\ \ldots\ldots,\ m\text{-}1),$

such that $\alpha(0)=0$, $\beta(0')=0'$ and $|\langle x_\sigma, y_\beta\rangle| = mn$.

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We call such pair (x_α, y_β) an admissible pair of $K_{m,n}$

and denote the corresponding embedding by $\mathfrak{M}(x_\alpha, y_\beta)$.

$x \leftrightarrow x_\alpha$ $y \leftrightarrow y_\beta$

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$\alpha^{-1}(-k) = -\alpha(k),\quad \beta^{-1}(-t') = -\beta(t')$

reflexible admissible pair of $K_{m,n}$

[Theorem]

1. For any (reflexible) edge transitive embedding $\mathfrak{M}$ of $K_{m,n}$, $\mathfrak{M}$ is isomorphic to $\mathfrak{M}(x_\alpha, y_\beta)$ for some (reflexible) admissible pair (x_α, y_β) of $K_{m,n}$.

2. For any admissible pairs (x_α, y_β), $(x_{\alpha'}, y_{\beta'})$ of $K_{m,n}$, $\mathfrak{M}(x_\alpha, y_\beta) \simeq \mathfrak{M}(x_{\alpha'}, y_{\beta'}) \Leftrightarrow \alpha = \alpha', \beta = \beta'$.

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Some observations

1. (x_α, y_β): admissible pair of $K_{m,n}$ $\Leftrightarrow$
$\forall g \in \langle x_\alpha, y_\beta \rangle$, $\exists\ i \in [n], j \in [m]$ s.t. $g = x_\alpha^i y_\beta^j$ $\Leftrightarrow$
$\forall\ i \in [n]$, $\exists\ a(i) \in [n], b(i) \in [m]$ s.t. $y_\beta x_\alpha^i = x_\alpha^{a(i)} y_\beta^{b(i)}$

Note that $a(i) = -\alpha^{-i}(-1)$ and $b(i) = \beta(i)$.

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2. (x_α, y_β): reflexible admissible pair of $K_{m,n} \Rightarrow$
$a(i) = -\alpha^{-i}(-1) = \alpha^i(1)$, $b(i) = \beta(i)$

3. (x_α, y_β): reflexible admissible pair of $K_{m,n}$, $d_1 = |\langle\alpha\rangle|, d_2 = |\langle\beta\rangle|$

$\Longrightarrow$ $\alpha(k) \equiv -k \pmod{d_2}$, $\beta(k) \equiv -k \pmod{d_1}$ $\Longrightarrow$

$\alpha(k+i) = \alpha^{(-1)^i}(k) + \alpha(i)$, $\beta(\mathrm{k+i}) = \beta^{(-1)^i}(k) + \beta(i)$ and

$d_1, d_2 \mid \gcd(m,n)$

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[Theorem]

(x_α, y_β): reflexible admissible pair of $K_{m,n}$ $\Leftrightarrow$

(1) $\alpha = id$, $\beta = id$ if m, n: odd;

(2) $\alpha(k) = kr$, $\beta(k) = k + (1 + (-1)^{k+1})s$ s.t.

(i) $\mathrm{r}^2 \equiv 1 \pmod{\mathrm{m}}$,

(ii) the smallest d satisfying $2ds \equiv 0 \pmod{n}$ divides gcd(m,n)

if m is odd and n is even;

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(3) $\alpha(2k) = 2kt_1$, $\alpha(2k+1) = 2kt_1 + 2s_1 + 1$

$\beta(2k) = 2kt_2$, $\beta(2k+1) = 2kt_2 + 2s_2 + 1$ s.t.

(i) d_1, $d_2 \mid \gcd(\mathrm{m,n})$, where $\mathrm{d}_1 = |\langle\alpha\rangle|$, $\mathrm{d}_2 = |\langle\beta\rangle|$.

(ii) $2\mathrm{t}_1^2 \equiv 2 \pmod{\mathrm{m}}$ and $2\mathrm{t}_2^2 \equiv 2 \pmod{\mathrm{n}}$.

(iii) $2(\mathrm{s}_1 + 1) \equiv 2(\mathrm{t}_1 + 1) \equiv 0 \pmod{\mathrm{d}_2}$ and $2(\mathrm{s}_2 + 1) \equiv 2(\mathrm{t}_2 + 1) \equiv 0 \pmod{\mathrm{d}_1}$

(iv) $2(\mathrm{s}_1 + 1)(\mathrm{t}_1 - 1) \equiv 0 \pmod{m}$ and $2(\mathrm{s}_2 + 1)(\mathrm{t}_2 - 1) \equiv 0 \pmod{n}$

if m, n:even.

[Theorem]

The number of reflexible edge transitive embedding of $K_{m,n}$ is

(1) 1 if m, n: odd

(2) $2^f(1+p_1^{c_1})\cdots(1+p_\ell^{c_\ell})$ if $m = p_1^{a_1}\cdots p_\ell^{a_\ell}p_{\ell+1}^{a_{\ell+1}}\cdots p_{\ell+f}^{a_{\ell+f}}$,

$n = 2^b p_1^{b_1}\cdots p_\ell^{b_\ell}q_{\ell+1}^{b_{\ell+1}}\cdots q_{\ell+g}^{b_{\ell+g}}$ and $\gcd(m,n)=p_1^{c_1}\cdots p_\ell^{c_\ell}$

(3) $A(a,b)2^{f+g+\ell}(1+p_1^{c_1})\cdots(1+p_\ell^{c_\ell})$ if $m = 2^a p_1^{a_1}\cdots p_\ell^{a_\ell}p_{\ell+1}^{a_{\ell+1}}\cdots p_{\ell+f}^{a_{\ell+f}}$,

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$$A(a,b) = \begin{cases} 1 & \textit{if } (a,b) = (1,1), \\ 2 & \textit{if } (a,b) = (1,2), \\ 4 & \textit{if } (a,b) = (2,2) \textit{ or } (1,k) \textit{ with } k \geq 3, \\ 10 & \textit{if } (a,b) = (2,3), \\ 12 & \textit{if } (a,b) = (2,k) \textit{ with } k \geq 4, \\ 28 & \textit{if } (a,b) = (3,3), \\ 40 & \textit{if } (a,b) = (3,4), \\ 36 & \textit{if } (a,b) = (3,k) \textit{ with } k \geq 5, \\ 20(1+2^{a-2}) & \textit{if } a = b \geq 4, \\ 20 + 18 \cdot 2^{a-2} & \textit{if } b-1 = a \geq 4, \\ 20 + 16 \cdot 2^{a-2} & \textit{if } b-2 \geq a \geq 4. \end{cases}$$

Product of two cyclic groups

$\Gamma = \langle x\rangle\langle y\rangle$ s.t. (i) $\langle x\rangle \cap \langle y\rangle = \{id\}$ (ii) $\langle x\rangle \simeq \mathbb{Z}_n$, $\langle y\rangle \simeq \mathbb{Z}_m$
(iii) $\exists\ \boldsymbol{\phi} \in Aut(\Gamma)$ s.t. $x^{\phi} = x^{-1}$, $y^{\phi} = y^{-1}$ $\Longrightarrow$
$\Gamma \simeq \langle x_\alpha, y_\beta\rangle$ for some reflexible admissible pair (x_α, y_β) of $\mathrm{K}_{m,n}$.

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$r^2 \equiv 1 \pmod m,\ \ s \equiv 0 \pmod{2^{b-1} q_{\ell+1}^{b_{\ell+1}} \cdots q_{\ell+g}^{b_{\ell+g}}}$, for $j = 1, \ldots, \ell$

$s \equiv 0 \pmod{\mathrm{p}_j^{b_j}}$ if $r \equiv 1 \pmod{\mathrm{p}_j^{a_j}}$, $s \equiv p_j^{b_j - c_j + z} \pmod{\mathrm{p}_j^{b_j}}$ if $r \equiv -1 \pmod{\mathrm{p}_j^{a_j}}$

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(3) if $m, n: even \Longrightarrow$

$$\Gamma \simeq \langle x, y \mid x^n = y^m = 1,\ yx = x^{2s_2+1}y^{2s_1+1},\ yx^2 = x^{2t_2}y^{2s_1(t_1+1)+1}, \\ y^2x = x^{2s_2(t_2+1)+1}y^{2t_1},\ y^2x^2 = x^2y^2\rangle$$

for some $s_1, t_1 \in [\frac{m}{2}]$, $s_2, t_2 \in [\frac{n}{2}]$ satisfying four conditions.

Future Work

1. Classifications of edge-transitive embeddings of $K_{m,n}$ and consequently classify group $\Gamma = \langle x \rangle \langle y \rangle$ s.t. (i) $\langle x \rangle \cap \langle y \rangle = \{id\}$ (ii) $\langle x \rangle \simeq \mathbb{Z}_n$, $\langle y \rangle \simeq \mathbb{Z}_m$.

2. Classifications of nonorientable edge-transitive embeddings of $K_{m,n}$.

Thank you!!!!