Automorphisms of Cayley Graphs that Respect Partitions

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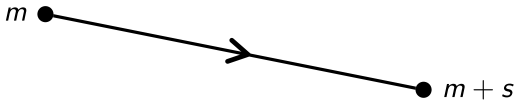

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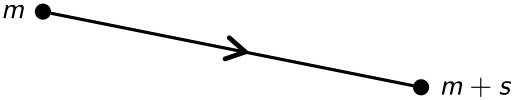
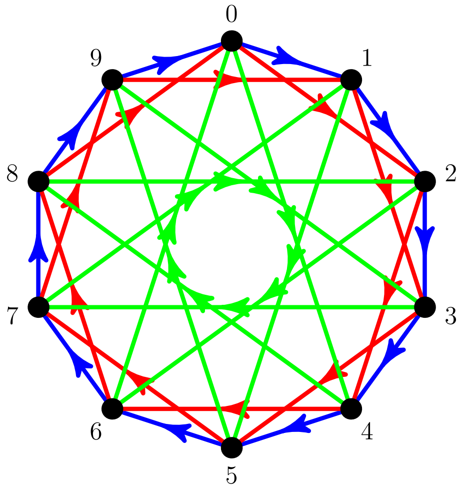


Example



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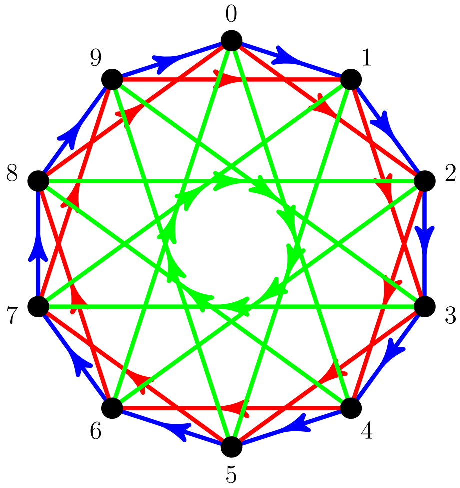

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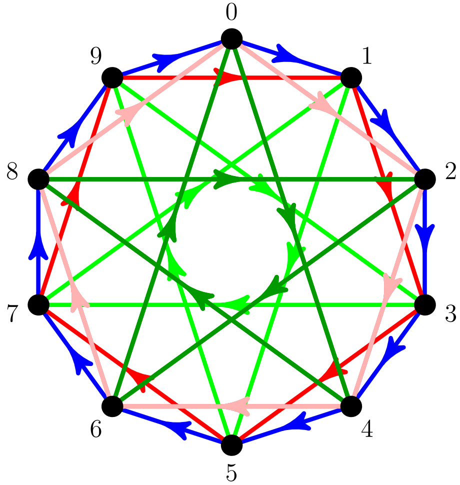

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We say that an automorphism of a graph respects *a partition of the edge set of the graph, if it takes every set of edges that is an element of the partition, to a set of edges that is also an element of the partition.*

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In a connected Cayley digraph $Cay(G; S)$, any automorphism α that respects the first partition and fixes the vertex 1, is an automorphism of G.

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We must show that for any $g, h \in G$, $\alpha(gh) = \alpha(g)\alpha(h)$. Since the graph is connected, we can write g as a word in $s_1, \ldots, s_k$.
What we will actually show is that for any $g = s_{i_1} \ldots s_{i_m}$, we must have $\alpha(g) = s_{\pi(i_1)} \ldots s_{\pi(i_m)}$; this is sufficient.

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So $\alpha(s_i s_j)$ could be any one of

- $s_{\pi(i)} s_{\pi(j)}$;
- $s_{\pi(i)} s_{\pi(j)}^{-1}$;
- $s_{\pi(i)}^{-1} s_{\pi(j)}$; or
- $s_{\pi(i)}^{-1} s_{\pi(j)}^{-1}$.

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Corollary

For circulant graphs (not just digraphs), a graph automorphism that respects the first partition and fixes the identity vertex, is necessarily a group automorphism.

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- (Easy consequence of definitions.) Any such $\beta\alpha$ fixes every coset of $\langle s \rangle$ setwise, for every $s \in S$.
- (With a lot of technical details.) If x, $x + s$, and $x + s'$ are all fixed by a graph automorphism that respects the second partition, then so is $x + s + s'$.

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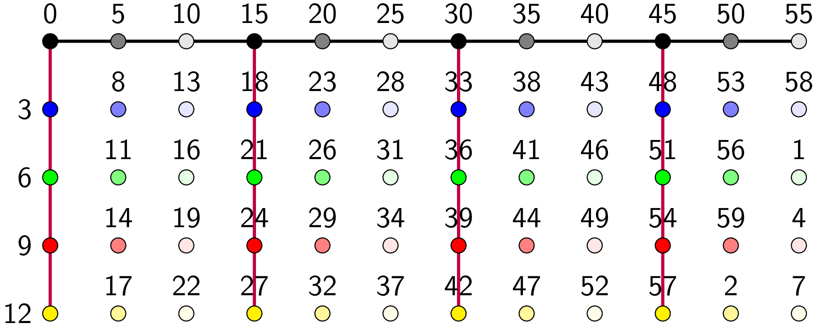

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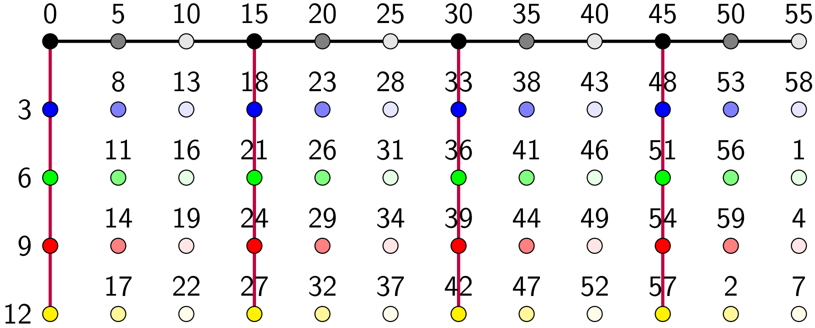

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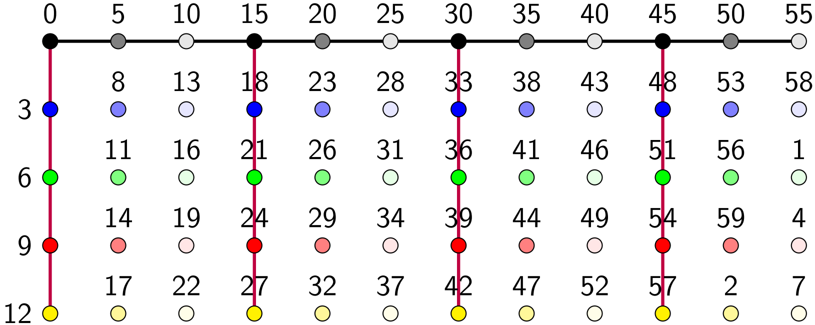

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But why pointwise? It turns out that if 8 moves to $8 + 15z$ with $0 < z < 4$, we can show that there is some prime that divides both $|3|/|15|$ and $|5|/|15|$, which is not possible.

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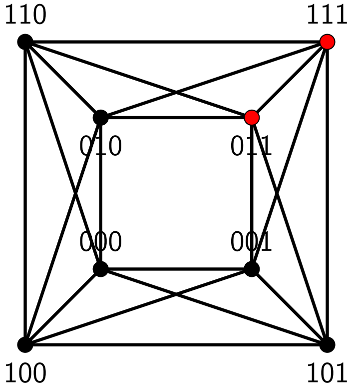

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Are there other natural partitions for which we could ask this question? E.g. edges that are mapped to one another by automorphisms of a vertex-transitive graph that is not a Cayley graph?

Thank you!