The 2-blocking number and the upper chromatic number of $\mathrm{PG}(2, q)$

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The problem

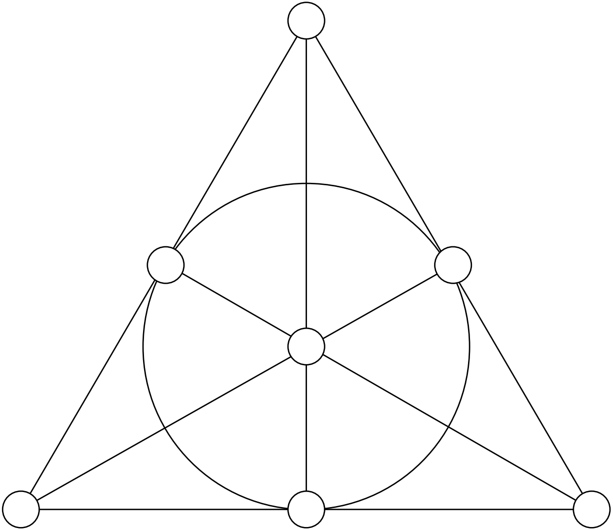
Color the vertices of a hypergraph $\mathcal{H}$.

A hyperedge is *rainbow*, if its vertices have pairwise distinct colors.

The upper chromatic number of $\mathcal{H}$, $\bar{\chi}(\mathcal{H})$: the maximum number of colors that can be used without creating a rainbow hyperedge (V. VOLOSHIN).

For graphs it gives the number of connected components. Determining $\bar{\chi}(\Pi_q)$ and $\bar{\chi}(\mathrm{PG}(2,q))$ has been a goal since the mid-1990s.

Example: $\bar{\chi}(\mathrm{PG}(2,2)) = 3$



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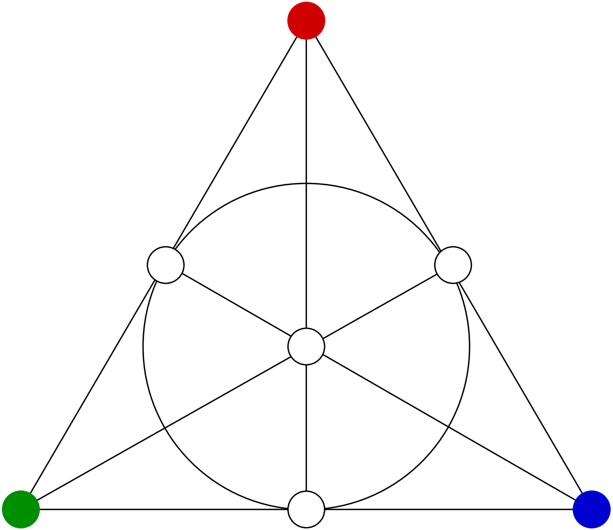

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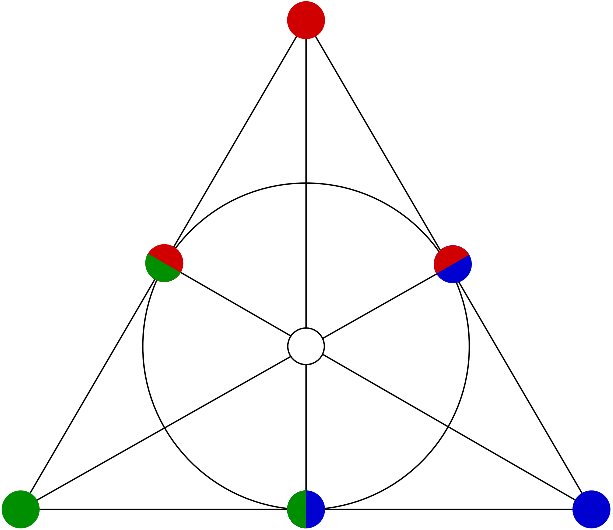

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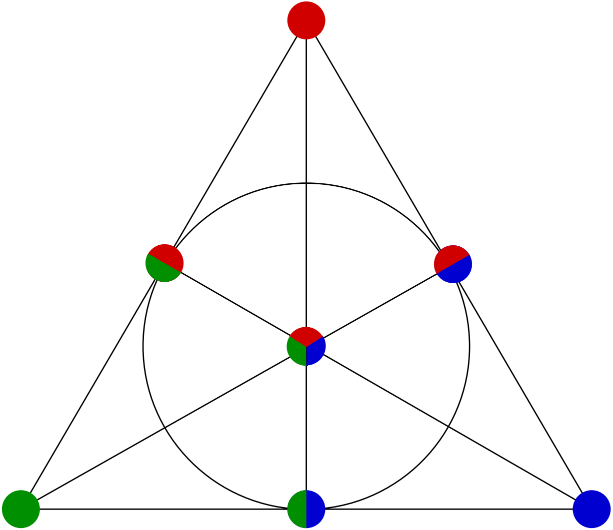

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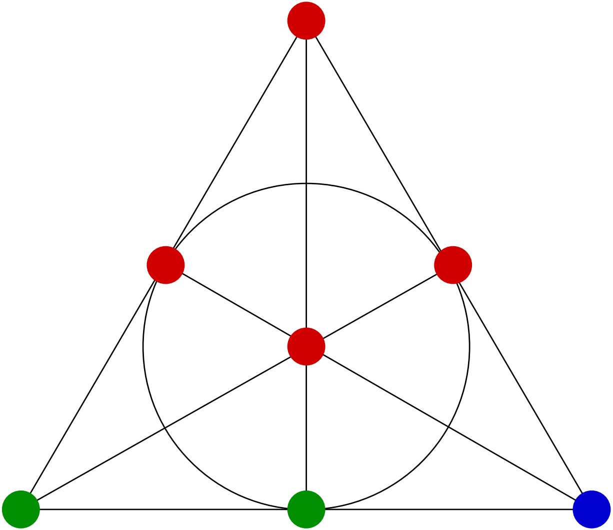
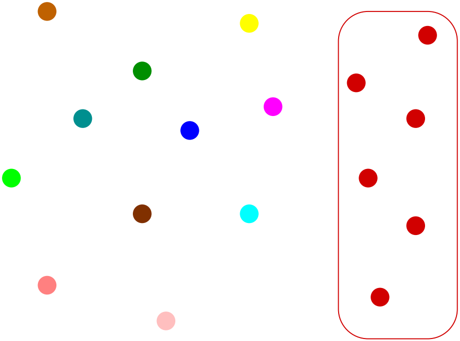


Trivial coloring



$v := q^2 + q + 1$, the number of points in Π_q.

$\tau_2 :=$ the size of the smallest double blocking set in Π_q.

Then $\bar{\chi}(\Pi_q) \geq v - \tau_2 + 1$.

We call this a *trivial coloring*.

What is known about blocking sets?

blocking set: meets every line, smallest one: line
non-trivial blocking set: contains no line
BRUEN: a non-triv. bl. set has $\geq q + \sqrt{q} + 1$ points, in case of equality it is a Baer subplane
Better results for $\mathrm{PG}(2, q)$, $q = p^h$, p prime:
BLOKHUIS for $q = p$, prime, the size is at least $3(p+1)/2$, and there are examples for every q
SzT, SZIKLAI: for $q \neq p$, a minimal blocking set meets every line in 1 modulo p (or rather in) 1 modulo p^e points with some $e|h$; there are several examples (linear bl. sets) In particular, there are bl. sets of size $q + ((q-1)/(p^e - 1)$ and $q + q/p^e + 1$.

What is known about (double) blocking sets?

double blocking set: meets each line in ≥ 2 pts. analogue of Bruen's bound: $|B| \geq 2q + \sqrt{2q} + ...$, not sharp

For PG$(2, q)$: $|B| \geq 2q + 2\sqrt{q} + 2$ (BALL-BLOKHUIS, sharp for q square. In case of equality: union of two Baer subplanes (GÁCS, SzT)

When q is prime, then $|B| \geq 2q + 2 + (q+1)/2$ (BALL. Known examples have at least $3p - 1$ points (examples are due to BRAUN, KOHNERT, WASSERMANN and recently to HÉGER).

The results are generalized to t-fold blocking sets, e.g. the lines meet small t-fold blocking sets in t modulo p points, see more details later.

What is known about τ_2?

Theorem

For the minimum size τ_2 of a double blocking set in $PG(2,q)$ the following is known:

1. *If q is a prime then $2q + (q+5)/2 \leq \tau_2 \leq 3q - 1$,*
2. *If q is a square then $\tau_2 = 2(q + \sqrt{q} + 1)$, and in case of equality the double blocking set is the union of two Baer subplanesm*
3. *If $q = p^h$, $h > 1$ odd, then $2q + c_p q^{2/3} \leq \tau_2 \leq 2(q + (q-1)/(p^e - 1))$, for the largest $e|h$, $e \neq h$.*

In (3), the lower and upper bounds have the same order of magnitude for $3|h$ (in particular, the lower bound can be improved to $2q + 2q^{2/3} - \ldots$, if $h = 3$). The upper bounds come from explicit constructions, e.g. by POLVERINO, STORME; see more details later.

Gábor Bacsó, Zsolt Tuza

Projective planes

Theorem (Bacsó, Tuza, 2007)

As $q \to \infty$,

- $\bar{\chi}(\Pi_q) \leq v - (2q + \sqrt{q}/2) + o(\sqrt{q})$;
- *for* q *square,* $\bar{\chi}(\mathrm{PG}(2,q)) \geq v - (2q + 2\sqrt{q} + 1) = v - \tau_2 + 1$;
- $\bar{\chi}(\mathrm{PG}(2,q)) \leq v - (2q + \sqrt{q}) + o(\sqrt{q})$;
- *for* q *non-square,* $\bar{\chi}(\mathrm{PG}(2,q)) \leq v - (2q + Cq^{2/3}) + o(\sqrt{q})$.

Theorem (Bacsó, Héger, SzT)

Let Π_q *be an arbitrary projective plane of order* $q \geq 4$, *and let* $\tau_2(\Pi_q) = 2(q+1) + c(\Pi_q)$. *Then*

$$\bar{\chi}(\Pi_q) < q^2 - q - \frac{2c(\Pi_q)}{3} + 4q^{2/3}.$$

Tamás Héger

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Improvement for projective planes

Theorem (Bacsó, Héger, SzT)

Let $v = q^2 + q + 1$*. Suppose that* $\tau_2(\mathrm{PG}(2,q)) \le c_0 q - 8$*,* $c_0 < 8/3$*, and let* $q \ge \max\{(6c_0 - 11)/(8 - 3c_0), 15\}$*. Then*

$$\bar{\chi}(\mathrm{PG}(2,q)) < v - \tau_2 + \frac{c_0}{3 - c_0}.$$

In particular, $\bar{\chi}(\mathrm{PG}(2,q)) \le v - \tau_2 + 7$*.*

Theorem (Bacsó, Héger, SzT)

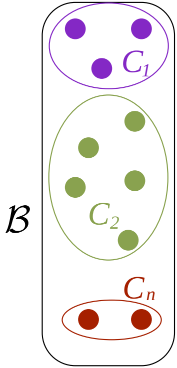
Let $q = p^h$*,* p *prime. Suppose that either* $q > 256$ *is a square, or* $h \ge 3$ *odd and* $p \ge 29$*. Then* $\bar{\chi}(\mathrm{PG}(2,q)) = v - \tau_2 + 1$*, and equality is reached only by trivial colorings.*

$C_1, \ldots, C_n$: color classes of size at least two (only these are useful)

C_i colors the line ℓ iff $|\ell \cap C_i| \geq 2$.

All lines have to be colored, so $\mathcal{B} = \bigcup_{i=1}^{n} C_i$ is a double blocking set.

We use $v - |\mathcal{B}| + n$ colors.



To reach the trivial coloring, we must have $v - |\mathcal{B}| + n \geq v - \tau_2 + 1$, thus we need

$$n \geq |\mathcal{B}| - \tau_2 + 1$$

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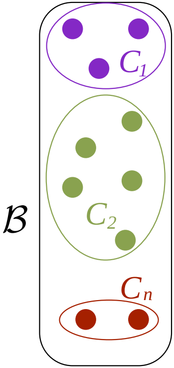

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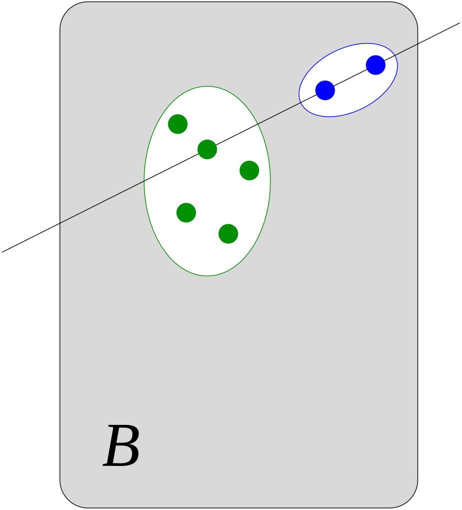
Eliminating color classes of size two



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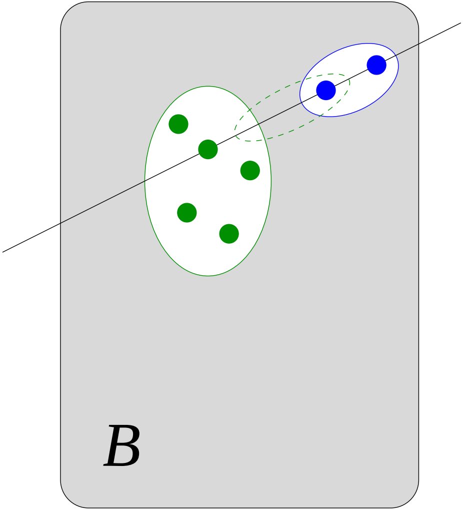

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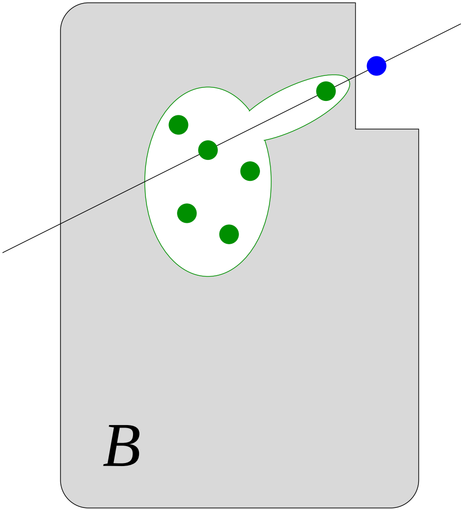

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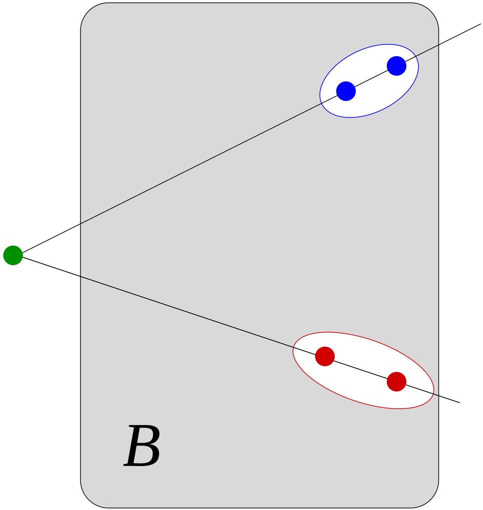

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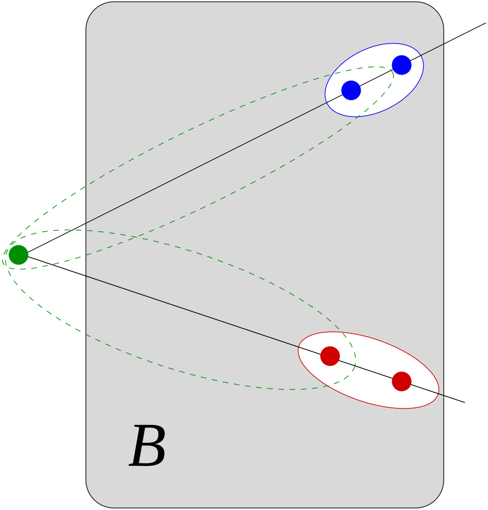

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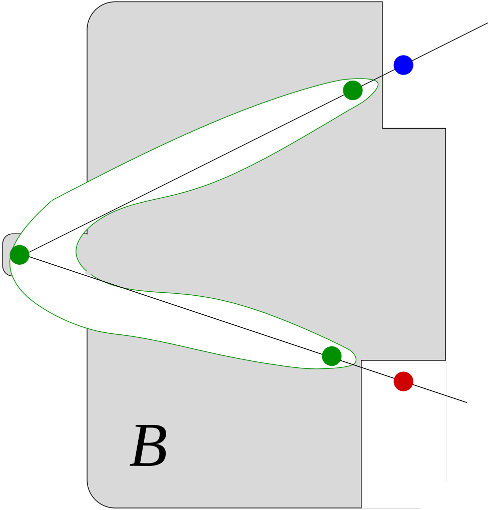

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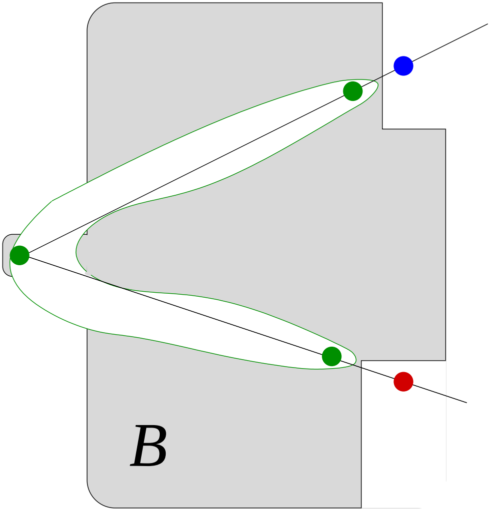

So there is at most one color class of size two.

Recall that $\tau_2 \lesssim 2.5q$.

$L(C_i) :=$ the number of lines colored by C_i. Then $L(C_i) \le \binom{|C_i|}{2}$.

By convexity, to satisfy

$$q^2 + q + 1 \le \sum L(C_i) \le \sum \binom{|C_i|}{2},$$

the best is to have one giant, and many dwarf color classes. But as

$$|\mathcal{B}| - \tau_2 + 1 \le n \le 1 + \frac{|\mathcal{B}| - \left|C_{\text{giant}}\right|}{3},$$

the giant can not be large enough.

$|\mathcal{B}| \gtrsim 3q - \varepsilon$

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However, if $\left|C_{\text{giant}}\right| \geq q + 2$, we use $L(C_i) \leq \frac{(q+1)}{2}|C_i|$.

Gács, Ferret, Kovács, Sziklai

Leo Storme

Zsuzsa Weiner

Lemma (Ferret, Storme, Sziklai, Weiner)

Let $\mathcal{B}$ be t-fold blocking set in $\mathrm{PG}(2,q)$, $|\mathcal{B}| = t(q+1)+k$, and $P \in \mathcal{B}$ be an essential point of $\mathcal{B}$. Then there are at least $(q+1-k-t)$ t-secants of $\mathcal{B}$ through P.

Corollary

Let $\mathcal{B}$ be a t-fold blocking set with $|\mathcal{B}| \leq (t+1)q$ points. Then there is exactly one minimal t-fold blocking set in $\mathcal{B}$, namely the set of essential points.

Remark

Harrach has a recent result on the unique reducibility of weighted t-fold $(n-k)$-blocking sets in the projective space $\mathrm{PG}(n,q)$.

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$\tau_2 + \varepsilon' \lesssim |\mathcal{B}| \lesssim 3q - \varepsilon$

Clear: if ℓ is a 2-secant to $\mathcal{B}$, then $\ell \cap \mathcal{B}$ is monochromatic.

Let $|\mathcal{B}| = 2(q+1) + k$. Then

Proposition

Every color class containing an essential point of $\mathcal{B}$ has at least $(q-k)$ points.

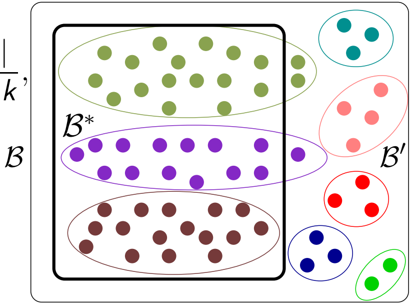
$\mathcal{B} = \mathcal{B}^* \cup \mathcal{B}'$, where $\mathcal{B}^*$ is the set of essential points, $|\mathcal{B}^*| \geq \tau_2$.

We have

$$|\mathcal{B}| - \tau_2 + 1 \leq n \leq \frac{|\mathcal{B}| - |\mathcal{B}^*|}{3} + \frac{|\mathcal{B}^*|}{q-k},$$

so

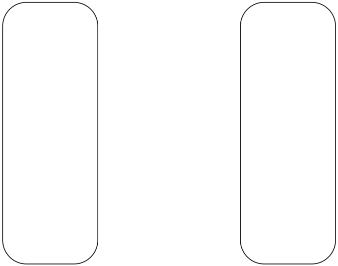
$$\frac{2}{3}(|\mathcal{B}| - \tau_2)(q-k) \leq \tau_2.$$



Aart Blokhuis

$|\mathcal{B}| \leq \tau_2 + \varepsilon$, $q > 256$ square (so $\tau_2 = 2(q + \sqrt{q} + 1)$)

Blokhuis, Storme, SzT: $\mathcal{B}$ contains two disjoint Baer subplanes, $\mathcal{B}_1$ and $\mathcal{B}_2$. $\mathcal{B}^* = \mathcal{B}_1 \cup \mathcal{B}_2$ can not be monochromatic.



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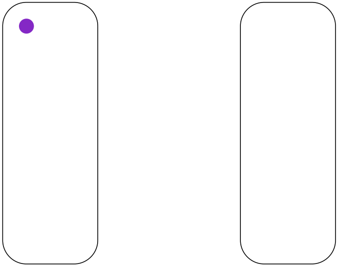

Let $P \in \mathcal{B}_1$ be purple.

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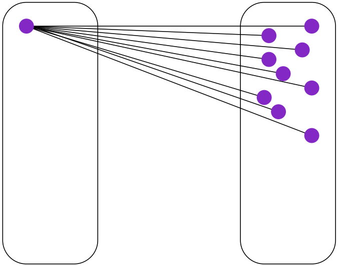

Let $P \in \mathcal{B}_1$ be purple. There are at least $(q - \sqrt{q} - \varepsilon - 1)$ 2-secants on P, so there are a lot of purple points in $\mathcal{B}_2$.

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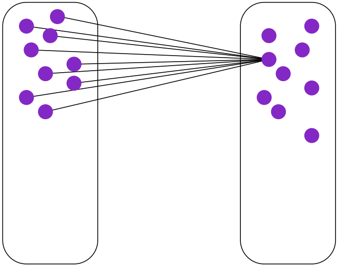

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The same from $\mathcal{B}_2$: we have at least $2(q - \sqrt{q} - \varepsilon - 1)$ purple points.

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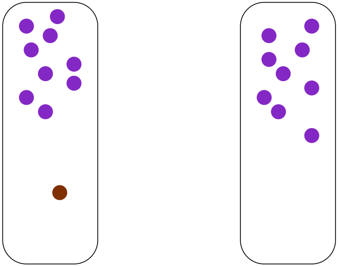

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If we have brown points as well:

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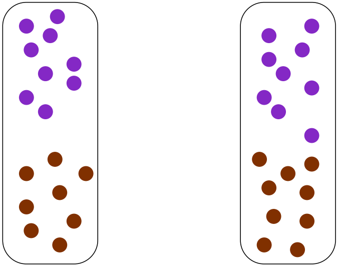

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The same from $\mathcal{B}_2$: we have at least $2(q - \sqrt{q} - \varepsilon - 1)$ purple points.

If we have brown points as well: $|\mathcal{B}| \geq 4(q - \sqrt{q} - \varepsilon - 1)$ $\lightning$

$|\mathcal{B}| \leq \tau_2 + \varepsilon$

By melting color classes, we may assume $n = 2$, $\mathcal{B}^* = \mathcal{B}^r \cup \mathcal{B}^g$, $|\mathcal{B}^*| = 2(q+1) + k$.

For a line ℓ, let

$$\begin{aligned} n_\ell^r &= |\mathcal{B}^r \cap \ell|, \\ n_\ell^g &= |\mathcal{B}^g \cap \ell|, \\ n_\ell &= n_\ell^r + n_\ell^g = |\mathcal{B} \cap \ell|. \end{aligned}$$

Define the set of red, green and balanced lines as

$$\begin{aligned} \mathcal{L}^r &= \{\ell \in \mathcal{L} \colon n_\ell^r > n_\ell^g\}, \\ \mathcal{L}^g &= \{\ell \in \mathcal{L} \colon n_\ell^g > n_\ell^r\}, \\ \mathcal{L}^= &= \{\ell \in \mathcal{L} \colon n_\ell^r = n_\ell^g\}. \end{aligned}$$

Using double counting, we get

$$\sum_{\ell \in \mathcal{L}} n_\ell = |\mathcal{B}^*|(q+1), \text{ hence}$$

$$\sum_{\ell \in \mathcal{L}:\ n_\ell > 2} n_\ell \geq \sum_{\ell \in \mathcal{L}} (n_\ell - 2) = |\mathcal{B}^*|(q+1) - 2(q^2+q+1) \gtrsim kq.$$

On the other hand, $\sum_{\ell \in \mathcal{L}:\ n_\ell > 2} n_\ell =$

$$\sum_{\ell \in \mathcal{L}^r:\ n_\ell > 2} (n_\ell^r + n_\ell^g) + \sum_{\ell \in \mathcal{L}^g:\ n_\ell > 2} (n_\ell^r + n_\ell^g) + \sum_{\ell \in \mathcal{L}^=:\ n_\ell > 2} (n_\ell^r + n_\ell^g) \leq$$

$$\sum_{\ell \in \mathcal{L}^r:\ n_\ell > 2} 2n_\ell^r + \sum_{\ell \in \mathcal{L}^g:\ n_\ell > 2} 2n_\ell^g + \sum_{\ell \in \mathcal{L}^=:\ n_\ell > 2} 2n_\ell^r \leq 4 \cdot \sum_{\ell \in \mathcal{L}^r \cup \mathcal{L}^=:\ n_\ell > 2} n_\ell^r.$$

Thus

$$\frac{kq}{4} \leq \sum_{\ell \in \mathcal{L}^r \cup \mathcal{L}^= :\ n_\ell > 2} n_\ell^r,$$

so there is a red point P with at least $\frac{kq}{4|\mathcal{B}^r|}$ (half)-red long secants through it.

Theorem (Blokhuis, Lovász, Storme, SzT)

Let B be a minimal t-fold blocking set in $\mathrm{PG}(2, q)$, $q = p^h$, $h \geq 1$, $|B| < tq + (q+3)/2$. Then every line intersects B in $t \pmod{p}$ points.

László Lovász

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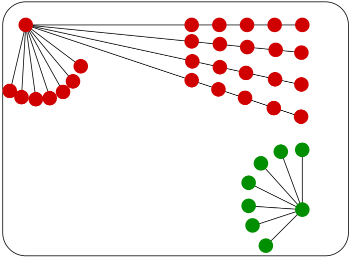
Thus on each of these long secants we find at least $p/2$ new red points.

$|\mathcal{B}| \leq \tau_2 + \varepsilon$

So we see:

$\frac{kpq}{8|\mathcal{B}^r|}$ red points on the red long secants through P,
$q - k$ red points on the red two-secants through P,
and $q - k$ green points.



Note that $|\mathcal{B}_r| \leq |\mathcal{B}| - |\mathcal{B}^g| \leq 2q + k - (q - k) = q + 2k < 2q$.

Thus $2q + k \gtrsim |\mathcal{B}| \geq 2q - 2k + \frac{kpq}{8|\mathcal{B}^r|} \geq 2q - 2k + \frac{kp}{16}$ $\lightning$

Two disjoint blocking sets

Let $q = p^h$, $h \geq 3$ odd, p not necessarily prime, p odd. Let $m = (q-1)/(p-1) = p^{h-1} + p^{h-2} + \ldots + 1$. Note that m is odd.

Let $f(x) = a(x^p + x)$, $a \in \mathrm{GF}(q)^*$. Then f is $\mathrm{GF}(p)$-linear, and determines the directions $\left\{\frac{f(x)-f(y)}{(x-y)} : x \neq y\right\}$ $=\{f(x)/x : x \neq 0\}=$ $\{(1 : f(x)/x : 0) : x \neq 0\}=$ $\{(x : f(x) : 0) : x \neq 0\}$. Thus

$$B_1 = \underbrace{\{(x : f(x) : 1)\}}_{A_1} \cup \underbrace{\{(x : f(x) : 0)\}_{x \neq 0}}_{I_1}$$

is a blocking set of Rédei type. Similarly, for $g(x) = x^p$,

$$B_2 = \underbrace{\{(y : 1 : g(y))\}}_{A_2} \cup \underbrace{\{(y : 0 : g(y))\}_{y \neq 0}}_{I_2}$$

is also a blocking set.

Two disjoint blocking sets

$$B_1 = \underbrace{\{(x : f(x) : 1)\}}_{A_1} \cup \underbrace{\{(x : f(x) : 0)\}_{x \neq 0}}_{l_1}$$

$$B_2 = \underbrace{\{(y : 1 : g(y))\}}_{A_2} \cup \underbrace{\{(y : 0 : g(y))\}_{y \neq 0}}_{l_2}$$

$f(x) = 0$ iff $x^p + x = x(x^{p-1} + 1) = 0$. As

$$-1 = (-1)^m \neq x^{(p-1)m} = x^{q-1} = 1,$$

$f(x) = 0$ iff $x = 0$.

$l_2 \cap B_1$ is empty, as $(0 : 0 : 1) \notin l_2$.

If $(x : f(x) : 0) \equiv (y : 1 : g(y)) \in l_1 \cap A_2$, then $g(y) = 0$, hence $y = 0$ and $x = 0$, a contradiction. So $l_1 \cap A_2 = \emptyset$.

Two disjoint blocking sets

$$B_1 = \underbrace{\{(x : f(x) : 1)\}}_{A_1} \cup \underbrace{\{(x : f(x) : 0)\}_{x \neq 0}}_{I_1}$$

$$B_2 = \underbrace{\{(y : 1 : g(y))\}}_{A_2} \cup \underbrace{\{(y : 0 : g(y))\}_{y \neq 0}}_{I_2}$$

Now we need $A_1 \cap A_2 = \emptyset$.

$(y : 1 : g(y)) \equiv (x : f(x) : 1)$ $(x \neq 0)$ iff

$(y; 1; g(y)) = (x/f(x); 1; 1/f(x))$, in which case

$1/f(x) = g(x/f(x)) = g(x)/g(f(x))$.

Thus we need that $g(x) = g(f(x))/f(x) = f(x)^{p-1}$ that is, $x^p = (a(x^p + x))^{p-1} = a^{p-1}x^{p-1}(x^{p-1} + 1)^{p-1}$ has no solution in $\mathrm{GF}(q)^*$.

Two disjoint blocking sets

Equivalent form:

$$\frac{1}{a^{p-1}} = \frac{(x^{p-1}+1)^{p-1}}{x} = (x^{p-1}+1)^{p-1}x^{q-2} =: h(x)$$

should have no solutions.

Let $D = \{x^m \colon x \in \mathrm{GF}(q)^*\} = \{x^{(p-1)} \colon x \in \mathrm{GF}(q)^*\}$. Then $1/a^{p-1} \in D$.

Note that $h(x) \in D \iff x \in D$.

So to find an element a such that $1/a^{(p-1)}$ is not in the range of h, we need that $h|_D \colon D \to D$ does not permute D.

Permutation polynomials

Theorem (Hermite-Dickson)

Let $f \in \mathrm{GF}(q)[X]$, $q = p^h$, p prime. Then f permutes $\mathrm{GF}(q)$ iff the following conditions hold:

- *f has exactly one root in $\mathrm{GF}(q)$;*
- *for each integer t, $1 \le t \le q-2$ and $p \not| t$, $f(X)^t \pmod{X^q - X}$ has degree $q-2$.*

A variation for multiplicative subgroups of $\mathrm{GF}(q)^*$:

Theorem

Suppose $d \mid q-1$, and let $D = \{x^d : x \in \mathrm{GF}(q)^*\}$ be the set of nonzero d^{th} powers. Assume that $g \in \mathrm{GF}(q)[X]$ maps D into D. Then $g|_D$ is a permutation of D if and only if the constant term of $g(x)^t \pmod{x^m - 1}$ is zero for all $1 \le t \le m-1$.

Two disjoint blocking sets

Recall that $h(X) = (X^{p-1}+1)^{p-1}X^{q-2}$. Let $t = p-1$, that is, consider

$$h^{p-1}(X) = \sum_{k=0}^{(p-1)^2} \binom{(p-1)^2}{k} X^{k(p-1)+(p-1)(q-2)} \pmod{X^m - 1}.$$

Since $k(p-1)+(p-1)(q-2) \equiv (k-1)(p-1) \pmod{m}$, the exponents reduced to zero are of form $k = 1 + \ell\frac{m}{(m,p-1)}$. Let r be the characteristic of the field $\mathrm{GF}(q)$. As $\binom{(p-1)^2}{1} \equiv 1 \pmod{r}$, it is enough to show that $\binom{(p-1)^2}{k} \equiv 0 \pmod{r}$ for the other possible values of k.

Suppose $h \geq 5$. Then $m/(m,p-1) > m/p > p^{h-2} > p^2$, thus by $k \leq (p-1)^2$, $\ell \geq 1$ does not occur at all. The case $h = 3$ can also be done.

Geertrui Van De Voorde

Two disjoint blocking sets

Using a higher dimensional representation of projective planes, VAN DE VOORDE could also contruct two disjoint bl. sets. Moreover she could specify that one of them is of Trace-type.

Theorem (G. Van De Voorde)

Let B be any non-trivial blocking set of size $< 3(q+1)/2$. Then there is a linear blocking set disjoint to B.

It is known that a GF(p^e)-linear blocking set $(e|h)$ has size at most $2(q+(q-1)/(p^e-1))$. Taking the smallest known blocking set (of size $q+q/p^e+1$) as B, it shows $\tau_2 \leq 2q+q/p^e+1+(q-1)/(p^e-1)$. She could also show the existence of a double blocking set of size $2(q+q/p^e+1)$.

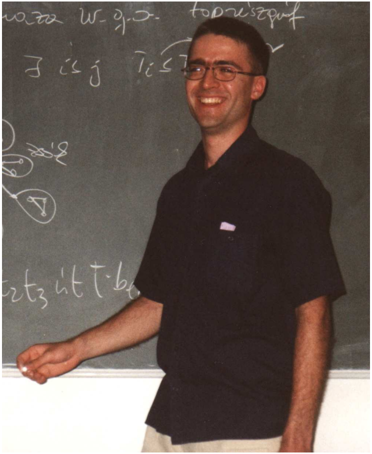
Multiple blocking sets in higher dims

Definition

A set B is a t-fold k-blocking set, if B meets each $(n-k)$-dim. subspace in $\geq t$ pts. In many cases B can be a multiset.

For $k = 1$ we just call them t-fold blocking sets. Trivial lower bound: $|B| \geq t(q+1)$ or $|B| \geq t(q^k + ... + q + 1))$ for k-blocking sets. In higher dims it can be reached as the sum (union) of lines (and similarly, if k is small, we have disjoint k-subspaces as the smallest examples).
Later we shall use results for $t = 2$. So $|B| \geq 2q^k + ...$ in this case.

János Barát



They prove similar results to the Blokhuis-Storme-SzT results in higher dims.

Theorem (Barát and Storme)

Let B be a t-fold 1-blocking set in $\mathrm{PG}(n,q)$, $q=p^h$, p prime, $q \geq 661$, $n \geq 3$, of size $|B| < tq + c_p q^{2/3} - (t-1)(t-2)/2$, with $c_2 = c_3 = 2^{-1/3}$, $c_p = 1$ when $p > 3$, and with $t < \min(c_p q^{1/6}, q^{1/4}/2)$. Then B contains a union of t pairwise disjoint lines and/or Baer subplanes.

The analogous result for (1-fold) blocking sets is due to STORME, WEINER. Some multiple points are also allowed here (in the plane).

t modulo p results

Theorem (Ferret, Storme, Sziklai, Weiner)

Let B be a minimal weighted t-fold blocking set in $\mathrm{PG}(2,q)$, $q = p^h$, p prime, $h \geq 1$, with $|B| = tq + t + k$, $t + k < (q+3)/2$, $k \geq 2$. Then every line intersects B in $t \pmod{p}$ points.

The corresponding result for non-weighted t-fold blocking sets is due to BLOKHUIS, LOVÁSZ, STORME, SzT. We remark that such a results (for non-weighted sets) immediately gives an upper bound on the size, which can be combined that the sizes are in certain intervals.

t modulo p result for k-blocking sets

Theorem (Ferret, Storme, Sziklai, Weiner)

A minimal weighted t-fold 1-blocking set B in $\mathrm{PG}(n, q)$, $q = p^h$, p prime, $h \geq 1$, of size $|B| = tq + t + k$, $t + k \leq (q-1)/2$, intersects every hyperplane in $t \pmod{p}$ points.

Theorem (Ferret, Storme, Sziklai, Weiner)

Let B be a minimal weighted t-fold $(n-k)$-blocking set of $\mathrm{PG}(n, q)$, $q = p^h$, p prime, $h \geq 1$, of size $|B| = tq^{n-k} + t + k'$, with $t + k' \leq (q^{n-k} - 1)/2$.
Then B intersects every k-dimensional subspace in $t \pmod{p}$ points.

Stability version of Ferret–Storme–Sziklai–Weiner

Theorem (Barát, Storme)

Let B be a t-fold 1-blocking set in $\mathrm{PG}(n, q)$, $q = p^h$, p prime, $q \geq 661$, $n \geq 3$, of size $|B| < tq + c_p q^{2/3}$, with $c_2 = c_3 = 2^{-1/3}$, $c_p = 1$ when $p > 3$, and with $t < c_p q^{1/6}/2$. Then B contains a union of t pairwise disjoint lines and/or Baer subplanes.

They also have more general results for k-blocking sets, but it is stated only for q square. As remarked earlier, the bounds for q non-square are signifivcantly weaker. Somewhat weaker but easier to prove bounds are due to KLEIN, METSCH. Recently, ZOLTÁN BLÁZSIK extended the results for q non-square.

The case of k-blocking sets

The case q square in the next theorem is due to FERRET, STORME, SZIKLAI, WEINER, the non-square case to BLÁZSIK.

Theorem (Ferret et al., Blázsik)

Let B be a t-fold k-blocking set in $\mathrm{PG}(n, q)$, $q = p^h$, p prime, $q \geq 661$, $n \geq 3$, of size $|B| = tq^k + c < tq^k + 2tq^{k-1}\sqrt{q} < tq^k + c_p q^{k-1/3}$, with $c_2 = c_3 = 2^{-1/3}$, $c_p = 1$ when $p > 3$, and with $t < c_p q^{1/6}/2$. Then B contains a union of t pairwise disjoint cones $\langle \pi_{m_i}, \mathrm{PG}(2k - m_i - 1, \sqrt{q}) \rangle$, $-1 \leq m_i \leq k - 1$, $i = 1, \ldots, t$

So, small enough double blocking sets can be decomposed into disjoint blocking sets.

Theorem (Héger-SzT)

Let $q \geq 37, n \geq 3$, $1 \leq k < n/2$ and consider $\Sigma = \mathrm{PG}_{n-k}(n, q)$. Then $\bar{\chi}(\Sigma) = v - \tau_2 + 1$. Moreover, if $d \leq q^k/20$ and $2d + 3 \leq c$, where c is the value in the stability result by Ferret-Storme-Sziklai-Weiner, then any proper coloring of Σ using at least $v - \tau_2 + 1 - d$ colors is trivial in the sense that it colors each point of two disjoint blocking sets with the same color.

Let $C_1, \ldots, C_m$ be the color classes of size at least two. Let $N = v - 2\left[{k+1 \atop 1}\right] + 1 - d$ be the no. of colors. As every $(n-k)$-space has to be colored, $B = C_1 \cup \ldots \cup C_m$ has to be a 2-fold k-blocking set.

Proposition

- $m \geq |B| - 2\left[{k+1 \atop 1}\right] + 1 - d$
- $m \leq 2\left[{k+1 \atop 1}\right] + d - 1$
- $|B| \leq 4\left[{k+1 \atop 1}\right] + 2(d-1)$

Sketch of the proof II

We say that a color class C *colors the* $(n-k)$*-space* U if $|C \cap U| \geq 2$.

Lemma

A color class C *colors at most* $\binom{|C|}{2}\begin{bmatrix} n-1 \\ k \end{bmatrix}$ *distinct* $(n-k)$*-spaces.*

Proposition

Let $q \geq 4$*, and suppose that* $d \leq 0.05q^k$*. Then* $|\mathcal{B}| \geq 4\begin{bmatrix} k+1 \\ 1 \end{bmatrix} - \sqrt{2}q^k + 2d + 2$ *cannot occur. In particular,* $|B| \geq (2.8 + 8/q)q^k + 2$ *cannot hold.*

Nóra Harrach

Sketch of the proof III

We need a result of NÓRA HARRACH: Suppose that a t-fold s-blocking set S in $\mathrm{PG}(n,q)$ has less than $(t+1)q^s + \begin{bmatrix} s \\ 1 \end{bmatrix}$ points. Then S contains a unique minimal t-fold s-blocking set S'.

Lemma

Suppose that a color class C contains an essential point P of B. Then C contains at least $3q^k - |B| + \begin{bmatrix} k \\ 1 \end{bmatrix}$ further essential points of $\mathcal{B}$, and for each such point Q there exists an $(n-k)$ space U such that $U \cap B = \{P; Q\}$. In particular, $|C| \geq 3q^k - |\mathcal{B}| + \begin{bmatrix} k \\ 1 \end{bmatrix} + 1$.

Proposition

Assume that $d \leq q^k/20$ and $q \geq 29$. Suppose that $|B| \leq 3q^k + \begin{bmatrix} k \\ 1 \end{bmatrix} - 4$; for example, $|B| \leq (2.8 + 8/q)q^k + 2$, where $q \geq 37$. Then $|B| \leq 2\begin{bmatrix} k+1 \\ 1 \end{bmatrix} + 2d + 3$.

Hence B is the union of two disjoint k-blocking set and they have to be colored by just one color.

Balanced upper chromatic number

ARAUJO-PARDO, KISS, MONTEJANO consider balanced coloring, when the sizes of color classes differ by at most 1. The maximum number of colors one can have in a balanced rainbow-free coloring is denoted by $\bar{\chi}_b$.

Theorem (Araujo-Pardo, Kiss, Montejano)

For a cyclic projective plane Π_q one has $((q^2+q+1)/6 \le \bar{\chi}_b(\Pi_q) \le (q^2+q+1)/3$, with equality in the upper bound if 3 divides q^2+q+1.

They also have some results for 3 and more dimensions and also results of similar flavour, see GYURI KISS's talk.

Thank you for your attention!