# COMPATIBLE ELEMENTS FOR A TRIDIAGONAL PAIR

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In this talk we will discuss relationships between the following:

- Tridiagonal pairs
- The Onsager algebra
- The $\mathfrak{sl}_2$ loop algebra
- Compatible elements for a tridiagonal pair

## *TRIDIAGONAL PAIRS*

Let $V$ denote a vector space over $\mathbb{F}$ with finite positive dimension. By a *tridiagonal pair* (or *TD pair*) on $V$ we mean an ordered pair $A, B$, where $A, B \in \mathrm{End}(V)$ satisfy the following four conditions:

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(1) each of $A, B$ is diagonalizable;

(2) there exists an ordering $\{V_i\}_{i=0}^{d}$ of the eigenspaces of $A$ such that $BV_i \subseteq V_{i-1} + V_i + V_{i+1}$ for $0 \leq i \leq d$, where $V_{-1} = 0$ and $V_{d+1} = 0$;

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(3) there exists an ordering $\{V'_i\}_{i=0}^{\delta}$ of the eigenspaces of $B$ such that $AV'_i \subseteq V'_{i-1} + V'_i + V'_{i+1}$ for $0 \leq i \leq \delta$, where $V'_{-1} = 0$ and $V'_{\delta+1} = 0$;

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(3) there exists an ordering $\{V_i'\}_{i=0}^{\delta}$ of the eigenspaces of $B$ such that $AV_i' \subseteq V_{i-1}' + V_i' + V_{i+1}'$ for $0 \leq i \leq \delta$, where $V_{-1}' = 0$ and $V_{\delta+1}' = 0$;

(4) there is no subspace $W$ of $V$ such that $AW \subseteq W$, $BW \subseteq W$, $W \neq 0$, $W \neq V$.

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- For $0 \leq i \leq d$ the spaces $V_i$, $V'_i$ have the same dimension; we denote this common dimension by $\rho_i$.
- The sequence $\{\rho_i\}_{i=0}^{d}$ is symmetric and unimodal; that is $\rho_i = \rho_{d-i}$ for $0 \leq i \leq d$ and $\rho_{i-1} \leq \rho_i$ for $1 \leq i \leq d/2$.

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- For $0 \leq i \leq d$ the spaces $V_i$, $V_i'$ have the same dimension; we denote this common dimension by $\rho_i$.
- The sequence $\{\rho_i\}_{i=0}^{d}$ is symmetric and unimodal; that is $\rho_i = \rho_{d-i}$ for $0 \leq i \leq d$ and $\rho_{i-1} \leq \rho_i$ for $1 \leq i \leq d/2$.
- It is also known that $\rho_i \leq \binom{d}{i}$ for $0 \leq i \leq d$.

## *TD PAIRS OF KRAWTCHOUK TYPE*

We say that a tridiagonal pair $A, B$ has *Krawtchouk type* whenever the eigenvalue corresponding to $V_i$ and $V'_i$ is $d - 2i$ for $0 \leq i \leq d$.

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In this case it is known that $A, B$ satisfy the Dolan-Grady relations

$$\begin{aligned} [A,[A,[A,B]]] &= 4[A,B], \\ [B,[B,[B,A]]] &= 4[B,A], \end{aligned}$$

where $[X, Y] = XY - YX$.

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where $[X, Y] = XY - YX$.

In view of these relations we consider the following Lie algebra.

## *THE ONSAGER ALGEBRA*

Let $\mathcal{O}$ denote the Lie algebra over $\mathbb{F}$ with generators $\mathcal{A}, \mathcal{B}$ and relations

$$\begin{aligned} [\mathcal{A},[\mathcal{A},[\mathcal{A},\mathcal{B}]]] &= 4[\mathcal{A},\mathcal{B}], \\ [\mathcal{B},[\mathcal{B},[\mathcal{B},\mathcal{A}]]] &= 4[\mathcal{B},\mathcal{A}]. \end{aligned}$$

We call $\mathcal{O}$ the *Onsager algebra*. We call $\mathcal{A}, \mathcal{B}$ the *standard generators* for $\mathcal{O}$.

Let $V$ denote a finite-dimensional irreducible $\mathcal{O}$-module. It turns out that the standard generators $\mathcal{A}, \mathcal{B}$ are diagonalizable on $V$. Furthermore there exist an integer $d \geq 0$ and scalars $\alpha, \beta \in \mathbb{F}$ such that the set of distinct eigenvalues of $\mathcal{A}$ (resp. $\mathcal{B}$) on $V$ is $\{d - 2i + \alpha | 0 \leq i \leq d\}$ (resp. $\{d - 2i + \beta | 0 \leq i \leq d\}$). We call the ordered pair $(\alpha, \beta)$ the *type* of $V$.

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Let $I$ denote the identity element of $\mathrm{End}(V)$. Replacing $\mathcal{A}, \mathcal{B}$ by $\mathcal{A} - \alpha I, \mathcal{B} - \beta I$ the type becomes $(0, 0)$.

## $\mathcal{O}$-MODULES and TD PAIRS

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Theorem (Hartwig):

Let $A, B$ denote a tridiagonal pair on $V$ of Krawtchouk type. Then there exists a unique $\mathcal{O}$-module structure on $V$ such that the standard generators $\mathcal{A}, \mathcal{B}$ act on $V$ as $A, B$ respectively. This $\mathcal{O}$-module is irreducible and of type $(0, 0)$.

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Theorem (Hartwig):

Let $V$ denote a finite-dimensional irreducible $\mathcal{O}$-module of type $(0, 0)$. Then the standard generators $\mathcal{A}, \mathcal{B}$ act on $V$ as a tridiagonal pair of Krawtchouk type.

Combining the previous two theorems we obtain a bijection between the following two sets:

(i) the isomorphism classes of tridiagonal pairs over $\mathbb{F}$ that have Krawtchouk type;

(ii) the isomorphism classes of finite-dimensional irreducible $\mathcal{O}$-modules of type $(0, 0)$.

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(i) the isomorphism classes of tridiagonal pairs over $\mathbb{F}$ that have Krawtchouk type;

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We will return to $\mathcal{O}$ shortly.

## *THE LIE ALGEBRA* $\mathfrak{sl}_2$

Let $\mathfrak{sl}_2$ denote the Lie algebra over $\mathbb{F}$ with basis $e, f, h$ and Lie bracket

$$[e, f] = h, \qquad [h, e] = 2e, \qquad [h, f] = -2f.$$

## *THE $\mathfrak{sl}_2$ LOOP ALGEBRA*

Let $t$ denote an indeterminate, and let $\mathbb{F}[t, t^{-1}]$ denote the $\mathbb{F}$-algebra consisting of the Laurent polynomials in $t$ that have all coefficients in $\mathbb{F}$. Let $L(\mathfrak{sl}_2)$ denote the Lie algebra over $\mathbb{F}$ consisting of the $\mathbb{F}$-vector space $\mathfrak{sl}_2 \otimes \mathbb{F}[t, t^{-1}]$ and Lie bracket

$$[u \otimes a, v \otimes b] = [u, v] \otimes ab, \quad u, v \in \mathfrak{sl}_2, \quad a, b \in \mathbb{F}[t, t^{-1}].$$

We call $L(\mathfrak{sl}_2)$ the $\mathfrak{sl}_2$ *loop algebra*.

## *AN EMBEDDING* $\mathcal{O} \to L(\mathfrak{sl}_2)$

E. Date and S. S. Roan showed that there exists a homomorphism of Lie algebras $\mathcal{O} \to L(\mathfrak{sl}_2)$ that sends

$$\begin{aligned} \mathcal{A} &\mapsto e \otimes 1 + f \otimes 1, \\ \mathcal{B} &\mapsto e \otimes t + f \otimes t^{-1}. \end{aligned}$$

Moreover, they showed that this map is injective.

## $\mathcal{O}$-modules and $L(\mathfrak{sl}_2)$-modules

In view of the above embedding $\mathcal{O} \to L(\mathfrak{sl}_2)$, we see that every $L(\mathfrak{sl}_2)$-module is an $\mathcal{O}$-module by restriction.

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E. Date and S. S. Roan showed that every finite-dimensional irreducible $\mathcal{O}$-module of type $(0, 0)$ can be obtained this way, as we shall explain shortly. We will discuss the various ways in which such an $\mathcal{O}$-module extends to an $L(\mathfrak{sl}_2)$-module, and we will discuss how these extensions are related to one another.

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The following theorem will help us describe these extensions.

Theorem:

The loop algebra $L(\mathfrak{sl}_2)$ is isomorphic to the Lie algebra over $\mathbb{F}$ that has generators $\mathcal{A}, \mathcal{B}, \mathcal{H}$ and relations

$$
\begin{aligned}
&[\mathcal{A}, [\mathcal{A}, \mathcal{H}]] = 4\mathcal{H}, && [\mathcal{H}, [\mathcal{H}, \mathcal{A}]] = 4\mathcal{A}, \\
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&[\mathcal{A}, [\mathcal{A}, [\mathcal{A}, \mathcal{B}]]] = 4[\mathcal{A}, \mathcal{B}], && [\mathcal{B}, [\mathcal{B}, [\mathcal{B}, \mathcal{A}]]] = 4[\mathcal{B}, \mathcal{A}], \\
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An isomorphism here is given by

$$\mathcal{A} \mapsto e \otimes 1 + f \otimes 1, \quad \mathcal{B} \mapsto e \otimes t + f \otimes t^{-1}, \quad \mathcal{H} \mapsto h \otimes 1.$$

## *COMPATIBLE ELEMENTS*

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For a TD pair $A, B$ on $V$ that has Krawtchouk type, an element $H \in \mathrm{End}(V)$ is said to be *compatible with $A, B$* whenever the following relations hold:

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\begin{aligned}
&[A,[A,H]] = 4H, && [H,[H,A]] = 4A, \\
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Let $\mathrm{Com}(A, B)$ denote the set of elements in $\mathrm{End}(V)$ that are compatible with $A, B$.

In the following two propositions, $V$ will denote a finite-dimensional irreducible $\mathcal{O}$-module of type $(0, 0)$. Let $A, B$ denote the tridiagonal pair associated with the $\mathcal{O}$-module $V$.

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Proposition: Consider an $L(\mathfrak{sl}_2)$-action on $V$ that extends the $\mathcal{O}$-action on $V$. For the $L(\mathfrak{sl}_2)$-module $V$, the action of $\mathcal{H}$ on $V$ is an element of $\mathrm{Com}(A, B)$.

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Proposition: Let $H \in \mathrm{Com}(A, B)$. Then there exists a unique $L(\mathfrak{sl}_2)$-action on $V$ that extends the $\mathcal{O}$-action on $V$, such that the element $\mathcal{H}$ of $L(\mathfrak{sl}_2)$ acts on $V$ as $H$.

## *Compatible elements and extensions*

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Combining these propositions we obtain a bijection between the following two sets:

(i) $\mathrm{Com}(A, B)$;

(ii) the $L(\mathfrak{sl}_2)$-actions on $V$ that extend the $\mathcal{O}$-action on $V$.

## *$\mathcal{O}$-modules and $L(\mathfrak{sl}_2)$-modules*

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To this end, we will recall the classification of $\mathcal{O}$-modules and $L(\mathfrak{sl}_2)$-modules. First we will summarize the classification of $L(\mathfrak{sl}_2)$-modules, which was proved by V. Chari. Then we will summarize the classification of $\mathcal{O}$-modules, which was proved by Date and Roan.

## *Evaluation modules for $L(\mathfrak{sl}_2)$*

There exists a family of $L(\mathfrak{sl}_2)$-modules called evaluation modules. Each evaluation module gets a notation of the form $V_d(a)$, where $d$ is a positive integer and $a$ is a nonzero scalar in $\mathbb{F}$.

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- On $V_d(a)$, each of the generators $\mathcal{A}, \mathcal{B}, \mathcal{H}$ is diagonalizable with eigenvalues $\{d-2i\}_{i=0}^{d}$.
- The $L(\mathfrak{sl}_2)$-module $V_d(a)$ is determined up to isomorphism by $d$ and $a$.

## A 2-DIMENSIONAL EXAMPLE

The actions of $\mathcal{A}, \mathcal{B}, \mathcal{H}$ on the $L(\mathfrak{sl}_2)$-module $V_1(a)$ are given by

$$\mathcal{A} : \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{B} : \begin{pmatrix} 0 & a \\ a^{-1} & 0 \end{pmatrix}, \quad \mathcal{H} : \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

with respect to a suitable basis.

## Irreducible $L(\mathfrak{sl}_2)$-modules (Chari):

Every finite-dimensional irreducible $L(\mathfrak{sl}_2)$-module is isomorphic to a tensor product of evaluation $L(\mathfrak{sl}_2)$-modules.

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A tensor product of evaluation $L(\mathfrak{sl}_2)$-modules

$$V_{d_1}(a_1) \otimes \cdots \otimes V_{d_n}(a_n)$$

is irreducible if and only if $a_1, a_2, \ldots, a_n$ are mutually distinct.

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Two such tensor products are isomorphic if and only if one can be obtained from the other by permuting the factors in the tensor product.

## *Inverse-free $L(\mathfrak{sl}_2)$-modules*

Let $V$ denote a finite-dimensional irreducible $L(\mathfrak{sl}_2)$-module. By the classification above it is isomorphic to a tensor product of evaluation modules $L(\mathfrak{sl}_2)$-modules, say

$$V_{d_1}(a_1) \otimes \cdots \otimes V_{d_n}(a_n).$$

$V$ is said to be *inverse-free* whenever $a_1, a_1^{-1}, a_2, a_2^{-1}, \ldots, a_n, a_n^{-1}$ are mutually distinct.

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$V$ is said to be *inverse-free* whenever $a_1, a_1^{-1}, a_2, a_2^{-1}, \ldots, a_n, a_n^{-1}$ are mutually distinct.

We are now ready to describe the relationship between $\mathcal{O}$-modules and $L(\mathfrak{sl}_2)$-modules more precisely.

*Irreducible $\mathcal{O}$-modules (Date and Roan):*

- Let $V$ denote a finite-dimensional irreducible $\mathcal{O}$-module of type $(0, 0)$. Then up to isomorphism $V$ is obtained by restricting the action of $L(\mathfrak{sl}_2)$ on a tensor product of evaluation $L(\mathfrak{sl}_2)$-modules.

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- Let $V$ denote a finite-dimensional irreducible $L(\mathfrak{sl}_2)$-module. When we restrict the action of $L(\mathfrak{sl}_2)$ to $\mathcal{O}$, the resulting $\mathcal{O}$-module is irreducible if and only if the $L(\mathfrak{sl}_2)$-module $V$ is inverse-free.

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- Let $V$ denote a finite-dimensional irreducible $L(\mathfrak{sl}_2)$-module. When we restrict the action of $L(\mathfrak{sl}_2)$ to $\mathcal{O}$, the resulting $\mathcal{O}$-module is irreducible if and only if the $L(\mathfrak{sl}_2)$-module $V$ is inverse-free.
- Two inverse-free tensor products of evaluation $L(\mathfrak{sl}_2)$-modules restrict to isomorphic $\mathcal{O}$-modules if and only if one can be obtained from the other by permuting the tensor factors and by replacing any number of the evaluation parameters with their multiplicative inverses.

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By the *degree* of a finite-dimensional irreducible $\mathcal{O}$-module of type $(0,0)$, we mean the number of tensor factors in the decomposition discussed above.

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(Remark: for a TD pair of Krawtchouk type, the degree equals $\rho_1$.)

We are now ready to discuss compatible elements in more detail.

## *Back to compatible elements . . .*

Until further notice, $A, B$ will denote a tridiagonal pair that has Krawtchouk type and degree $n$.

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Proposition:

- The set $\mathrm{Com}(A, B)$ has cardinality $2^n$.

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We identify the underlying vector space $V$ with an irreducible, inverse-free $L(\mathfrak{sl}_2)$-module

$$V_{d_1}(a_1) \otimes \cdots \otimes V_{d_n}(a_n),$$

such that $\mathcal{A}, \mathcal{B}, \mathcal{H}$ act on $V$ as $A, B, H$ respectively.

For $1 \leq i \leq n$ let $\mathcal{H}_i \in \mathrm{End}(V)$ be

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Proposition: The set $\mathrm{Com}(A, B)$ consists of the elements

$$\sum_{i=1}^{n} \varepsilon_i \mathcal{H}_i \qquad \varepsilon_i \in \{\pm 1\}, \quad 1 \leq i \leq n.$$

## *SPECIAL CASE:* $\rho_i = \binom{d}{i}$

We now consider a special case when the description of $\mathrm{Com}(A, B)$ is especially nice. We assume that $\rho_i = \binom{d}{i}$ for $0 \leq i \leq d$, where $d$ is the diameter of $A, B$.

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(ii) For each $x \in \mathbb{X}$ there exists $H_x \in \mathrm{Com}(A, B)$ such that for $0 \leq i \leq d$, the sum of of the elements in $\mathbb{X}$ at distance $i$ from $x$ is an eigenspace for $H_x$ with eigenvalue $d - 2i$.

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So the elements of $\mathrm{Com}(A, B)$ correspond to the $2^d$ dual adjacency matrices for the $d$-cube.

*Picture for $H_x$ when $d = 3$:*