On the number of words of a given GC-content in some cyclic DNA-codes

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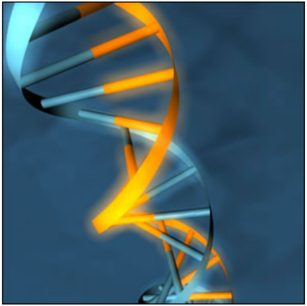
Biological preliminaries



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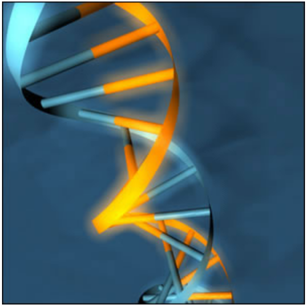

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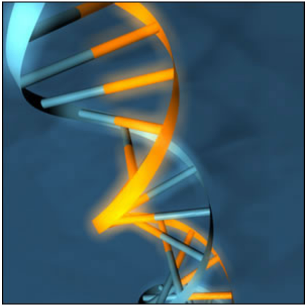

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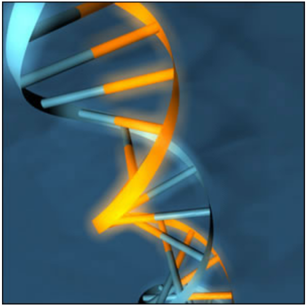

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$A \longleftrightarrow T, C \longleftrightarrow G$

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Thus, the complete sequence of a gene determines the sequence of aminoacids of a protein.

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4. Cosets of linear codes

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Theorem

A cyclic code C over $\mathbb{F}_4$ is complementable if and only if $X - 1$ does not divide the generator polynomial of the code C.

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If q is a prime power and $g(X) = g_0 + \cdots + X^r \in \mathbb{F}_q[X]$ is a monic polynomial of degree r dividing $X^n - 1$, then the reciprocal polynomial of $g(X)$ is the polynomial $g_R(X) = g_0^{-1} X^r g(X^{-1})$. The polynomial $g(X)$ is called self-reciprocal if $g(X) = g_R(X)$.

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Theorem (Massey)

A cyclic code C over $\mathbb{F}_4$ is reversible if and only if the generator polynomial of the code C is self-reciprocal.

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4. GC-content constraint: the number of positions in which a nucleotide G or C appears is the same for all the words of the code.

Combinatorial restrictions on the words of a DNA-code

As usual in the literature of DNA-codes, $\max_w A_4^{GC,RC}(n,d,w)$ will denote the maximum number number of words in a DNA-code of length n satisfying the Hamming constraint and the reverse-complement constraint with parameter d and the constant GC-content constraint.

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It is well known that, if C is a complementable reversible DNA-code with minimum distance d, we can put $C = C_0 \cup C_1 \cup C_2$, where C_0 is the set of words in C which coincide with their reverse complement and where $u^{rc} \in C$ if and only if $u \in C$, and $\max_w A_4^{GC,RC}(n, d, w) \geq |C_1|$.

Number of words with a given GC-content

Definition

Let $u \in \mathbb{F}_4^n$. We will call $\mathbb{F}_2$-weight of u, and we will denote it $wt_{\mathbb{F}_2}(u)$, to the number of coordinates of u which are in $\mathbb{F}_2$. If $C \subseteq \mathbb{F}_4^n$ is a code over $\mathbb{F}_4$, we define the $\mathbb{F}_2$-weight enumerator polynomial to be

$$W_{\mathbb{F}_2,C}(X) = \sum_{u \in C} X^{wt_{\mathbb{F}_2}(u)} = \sum_{w \geq 0} b_w X^w,$$

where $b_w = b_w(C)$ is the number of words of the code C with $\mathbb{F}_2$-weight equal to w.

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Let $g \in \mathbb{F}_4[X]$ *be a divisor of* $X^n - 1$. *We will say that the cyclic code generated by* g *is Galois-supplemented if* $(g, g^\sigma) = 1$, *where* σ *is the Frobenius automorphism of* $\mathbb{F}_4$ *over* $\mathbb{F}_2$.

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Theorem

Let $C \subseteq \mathbb{F}_4^n$ be a Galois-supplemented cyclic code with generator polynomial g. Then,

$$W_{F_2,C}(X) = 2^{n-2\deg g}(X+1)^n.$$

Number of words with a given GC-content

Aboulion et al. gave a table of lower bounds for $\max_w A_4^{GC,RC}(n,d,w)$ for $n \leq 30$. In particular, for $\max_w A_4^{GC,RC}(29,11,w)$ they obtained the bound 38777664. By considering the Quadratic-residue code of length 29 over $\mathbb{F}_4$, which is complementable and reversible, and whose minimum distance is 11, and using the previous Theorem, we obtain that $\max_w A_4^{GC,RC}(29,11,w) \geq 77558760$, and so we have improved that bound for this set of parameters.

Future research

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2. Take advantage of good symmetry groups on codes.

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THANK YOU VERY MUCH FOR YOUR ATTENTION!