Orientably-Regular Embeddings of Graphs of Order Prime-Cube

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1. Definitions

Surfaces and Embeddings

*Surface S*: closed, connected 2-manifold;

*Classification of Surfaces*:

(i) Orientable Surfaces: $S_g$, $g = 0, 1, 2, \cdots$,
$$v + f - e = 2 - 2g$$

(ii) Nonorientable Surfaces: $N_k$, $k = 0, 1, 2, \cdots$,
$$v + f - e = 2 - k$$

*Embeddings* of a graph $X$ in the surface is a continuous one-to-one function $i : X \rightarrow S$.

*2-cell Embeddings*: each region is homeomorphic to an open disk.
**Topological Map** $\mathcal{M}$: a 2-cell embedding of a graph into a surface. The embedded graph $X$ is called the *underlying graph* of the map.

**Automorphism** of a map $\mathcal{M}$ : an automorphism of the underlying graph $X$ which can be extended to self-homeomorphism of the surface.

**Orientation-Preserving Automorphism** of an orientably map $\mathcal{M}$ : an automorphism of Preserving Orientation of the map

**Automorphism group** $\text{Aut}(\mathcal{M})$ : all the automorphisms of the map $\mathcal{M}$.

**Orientation-preserving automorphisms group** $\text{Aut}^+\mathcal{M}$ of $\mathcal{M}$: all the orientation-preserving automorphism.
Flag: incident vertex-edge-face triple

Arc: incident vertex-edge pair

Remark: $\text{Aut}(\mathcal{M})$ acts semi-regularly on the flags of $X$.

Remark: $\text{Aut}^+ (\mathcal{M})$ acts semi-regularly on the arcs of $X$. 
Regularity of Maps

Regular Map: $\text{Aut}(\mathcal{M})$ acts regularly on the flags.

Orientably Regular Map: $\text{Aut}^+(\mathcal{M})$ acts regularly on the arcs.

Reflexible Map: Orientably Regular, admitting orientation-reversing automorphisms

Chiral Map: Orientably Regular, without any orientation-reversing automorphisms
Regular Map
= Nonorientably Regular Map
∪ Reflexible Orientably Regular Map

Orientably Regular Map
= Reflexible Orientably Regular Map
∪ Chiral Orientably Regular Map
Combinatorial and Algebraic Map

Combinatorial Orientably Map:

graph $X = (V, D)$, with vertex set $V = V(X)$, dart (arc) set $D = D(X)$.

*arc-reversing involution* $L$: interchanging the two arcs underlying every given edge.

*rotation* $R$: cyclically permutes the arcs initiated at $v$ for each vertex $v \in V(X)$.

*Map $\mathcal{M}$ with underlying graph $X$*: the triple $\mathcal{M} = \mathcal{M}(X; R, L)$. 
Remarks:

Monodromy group $\text{Mon}(\mathcal{M}) := \langle R, L \rangle$ acts transitively on $D$.

Given two maps

$$\mathcal{M}_1 = \mathcal{M}(X_1; R_1, L_1), \quad \mathcal{M}_2 = \mathcal{M}(X_2; R_2 L_2),$$

Map isomorphism: bijection $\phi : D(X_1) \rightarrow D(X_2)$ such that

$$L_1 \phi = \phi L_2, \quad R_1 \phi = \phi R_2$$

Automorphism $\phi$ of $\mathcal{M}$: if $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}$;

Automorphism group: $\text{Aut}(\mathcal{M})$
Algebraic Orientably Maps:

Orientably Regular Map:

\[ G = \text{Aut}(\mathcal{M}) = \langle r, l \rangle \cong \text{Mon}(\mathcal{M}) = \langle R, L \rangle \]

\[ \mathcal{M} = \mathcal{M}(G; r, l) \]

\[ D = G, \quad \text{Mon}(\mathcal{M}) = L(G), \quad \text{Aut}(\mathcal{M}) = R(G) \]

the orbits of \( \langle r \rangle, \langle l \rangle \) and \( \langle rl \rangle \) are vertices, edges and faces, with the natural inclusion relation
ORM without multiple edges

A regular map with multiple edges projects onto another one with a simple underlying graph that has the same set of vertices and the same adjacency relation.

Regular maps with multiple edges can be described as some “extensions” of regular embeddings of simple graphs.
\[ G = \langle r, \ell \rangle \rightarrow \]

\[ \overline{G} = G/K, \quad K = \langle r \rangle_G — \text{core of } \langle r \rangle \text{ in } G \]

To determine the ORM with multiple edges from an ORM of a simple graph is essentially a group cyclic extension problem.

*Here we just consider the ORM without multiple edges*
2. Regular maps of given order

Set \( v = \text{order of graphs} \)

ORM with given order \( v \) \( \Leftrightarrow \) ORM with given graphs

because one may pick up the symmetric graphs of order \( v \) with a arc-regular subgroup.
1. \( v = p \) — a prime:

\[
G = \langle a, b \mid a^q = b^s = 1, a^b = a^t \rangle,
\]

\[
\mathcal{M} = \mathcal{M}(G; b^e, (b^s)^a), \quad e \in \mathbb{Z}_s^*.
\]

S.F. Du, J.H. Kwak and R. Nedela, Regular embeddings of complete multipartite graphs, *EJC* 26(2005), 437–452
2. $v = pq$ = a product of two primes


3. \( v = p^3 \):

*Motivations:*

(a) To understand permutation groups of degree \( p^3 \) in more details (subgroup structure) is still a difficult problem.

(b) Classification for symmetric graphs of order \( p^3 \) is still open.
(c) Complete classification for semi-symmetric symmetric graphs order $2p^3$ is still open,

only partial results are given, that is $\text{Aut}(X)$ acts infaithfully on one bipart, see

(i) L. Wang, S.F. Du, X.W. Li, A Class of Semisymmetric Graphs, 
$AMC$, 7 (2014) 40 C53

(ii) L. Wang, S. F. Du, SEMISYMMETRIC GRAPHS OF ORDER $2p^3$, $EJC$, 36 (2014) 393 C405


$\text{Aut}(X)$ acts faithfully on each bipart: that is a hard part for this work.
(d) For ORM of order $p^3$, we may do that, because we do have a particular subgroup of degree $p^3$, that is $\text{Aut}(\mathcal{M}) = \langle r, \ell \rangle$, which is an arc-regular subgroup of the graph.

(e) Many recent results on ORM can help us to do this work.
Notation:

$\Gamma$ = a connected simple graph of order $p^3$ where $p$ is prime and of valency $n$

$\mathcal{M}$ = an ORM of $\mathcal{G}$

$G = \langle r, \ell \rangle$ = the orientation preserving group of $\mathcal{M}$

$\ell^2 = 1, \langle r \rangle = G_v$ for a vertex $v$ in $V(\Gamma)$.

$P$ = a Sylow subgroup of $G$

$N$ = a minimal normal subgroup of $G$

$\mathcal{B}$ = the orbits of $N$ on the vertices

$K$ = be the kernel of $G$ acting on $\mathcal{B}$ and $\overline{G} = G/K$. 
3.1 Group structure for $G$

**Theorem**

1. $|P| = p^3, p^4$ or $p^5$.
2. $G = P \rtimes \langle r^m \rangle$ where $m = |\langle r \rangle \cap P|$.
3. $N = \mathbb{Z}_p^k$, $k = 1, 2, 3$, and either
   - (3.1) $N$ is transitive on $V$ and $G$ is a primitive affine group; or
   - (3.2) $N$ induce a blocks of length $p$ such that $N \cong \mathbb{Z}_p \leq Z(P)$ and either $K \cong \mathbb{Z}_p \rtimes \mathbb{Z}_t$ for some $t \in \mathbb{Z}_p^*$; or $K \cong \mathbb{Z}_p^2$. 

Remark: From $G = P \rtimes \langle r^m \rangle$, we need to

study the split cyclic extension of $P$ by $\mathbb{Z}_{n_1}$ where $n = mn_1$ for $m = P \cap \langle r \rangle$ and $m = 1, p, p^2$, $(n_1, p) = 1$.

$\iff$

to determine the conjugacy classes of cyclic subgroups of order prime to $p$ in $\text{Aut}(P)$, noting that $|P| = p^3, p^4$ or $p^5$.
This case is quite complicated. Fortunately, it becomes more easy, because we may employ many known results!

$|P| = p^5 \implies \Gamma$ is a $p$-partite graph such that any two connected biparts is complete bipartite graph.
Recalling some known results:

\[ K_m[nK_1] = \text{the complete multipartite graph with } m \text{ parts, while each part contains } n \text{ vertices.} \]

(i) \( m = 1 \): Complete graphs:

ORM:


NORM:

S. E. Wilson, Cantankerous maps and rotary embeddings of \( K_n \), *JCTB* 47 (1989), 262–273.
(ii) $m = 2$ : Complete bipartite graphs $K_{2n}K_1 = K_{n,n}$:

ORM:


$n = p^k$, $p$ is odd prime:
G.A. Jones, R. Nedela and M. Škoviera, Regular embeddings of $K_{n,n}$ where $n$ is an odd prime power, *EJC* 28(2007), 1863-1875.

$n = 2^k$,

Any $n$:

Other partial results:


NORM:

(iii). \( m \geq 3 \): Complete multipartite graphs \( K_{m[n]} \):


\( m \geq 3 \) and \( n \geq 2 \):


General question:

For any connected graph $X$ of order $m$, let $X[nK_1]$ be the $m$-partite graph, while each part contains $n$ vertices and the block graph induced by the partition is isomorphic to $X$. Suppose that $X$ has a RM. Classify the RM of $X[nK_1]$. 
$X$ is of prime order:


This paper depends heavily on classification of ORM of $K_m[nK_1]$ mentioned as above.
Theorem

Suppose that $|P| = p^5$. Then $G$, $\mathcal{M}$ and the genus $g$ are given by

(1) $p = 2$, $n = 4$:

$$G_1 \cong \langle a, b, x | a^4 = b^4 = x^2 = 1, [a, b] = 1, a^x = b \rangle,$$

$$\mathcal{M}_1 = \mathcal{M}(G_1; a, x), \quad g = 3.$$

(2) $p = 2$, $n = 4$:

$$G_2 \cong \langle a, b, x | a^4 = b^4 = x^2 = 1, [b, a] = a^2 b^2, \quad [a^2, b] = [b^2, a] = 1, a^x = b \rangle,$$

$$\mathcal{M}_2 = \mathcal{M}(G_2; a, x), \quad g = 1.$$
(3) $p = 3, n = 18$:

\[ G_3 \cong \langle a, b | a^{18} = b^2 = c^{27} = 1, c = a^9 b, c^a = c^2 \rangle, \]

\[ M_3(j) = M(G_3; a^j, b) \text{ where } j \in \mathbb{Z}^*_1, \quad g = 397. \]

(4) $p = 3, n = 18$:

\[ G_4(i, j) \cong \langle a, b | a^{18} = b^2 = 1, a^2 = x, x^b = y, [x, y] = x^{3i}y^{-3i}, \]

\[ y^a = x^{-1}y^{-1}, (ab)^3 = x^{3j}y^{-3j} \rangle, \]

where $(i, j) = (0, 0), (0, 1), (1, 0), (1, 1)$ or $(1, -1)$;

\[ M_4(i, j, l) = M(G_4(i, j); a^l, b), \]

where $l = 1$ for $(i, j) = (0, 0)$ and $l = \pm 1$ for the other cases.

$g = 55$ for $(i, j, l) = (0, 0, 1)$ and $(1, 0, \pm 1)$;

$g = 163$ for $(i, j, l) = (0, 1, \pm 1)$, and $(1, \pm 1, \pm 1)$. 
(5) $p \geq 5$, $n = p^2 s$, $s$ is a even divisor of $p - 1$ and $e$ is of order $sp^2$ in $\mathbb{Z}_{p^3}^*$:

$$G_5(p, s) \cong \langle a, x \mid a^{sp^2} = x^{p^3} = 1, a^x = a^e \rangle,$$

$$M_5(p, s, j) = M(G_1; a^j, a^{\frac{p^2 s}{2}} c) \quad \text{where} \quad j \in \mathbb{Z}_{p^2 s}^*,$$

$$g = 1 + \frac{1}{4} p^3 (sp^2 - 4) \quad \text{for} \quad 4 \mid s; \quad g = 1 + \frac{1}{4} p^3 (sp^2 - 4) \quad \text{for} \quad 4 \nmid s.$$

Moreover, the above groups and maps are uniquely determined by the given parameters.
Suppose that $|P| = p^3$. Then $G$ and $M$ are given by

(1) Define three affine subgroups and the corresponding maps:

(1.1) $G_{11}(p, n) = T : \langle x \rangle$,

where $x = ||e, d\lambda, f\lambda; f, e + d\varepsilon, f\varepsilon + d\lambda; d, f, e + d\varepsilon||$,

where $p \geq 2$, $n \mid p^3 - 1$ but $n \nmid p^2 - 1$; and $e + f\beta + d\beta^2$ is a fixed element of order $n$ in $\mathbb{F}_{p^3}^*$.

$M_{11}(p, n, i, j) = M(G_{11}(p, n); x^i, t_{(1, 0, 0)}x^{jn/2})$,

where $i \in \mathbb{Z}_n^*/(\mathbb{Z}_n^*)^{3+}$, $j = 0$ for $p = 2$, and $j = 1$ for $p \geq 3$. 
\( G_{12}(p, h, d) = T : \langle x \rangle, \)

\[
x = ||1, 1, 0; 1, 0, 0; 0, 0, 1|| \quad \text{for } p = 2 \text{ and } n = 3;
\]

\[
x = ||e, f \theta, 0; f, e, 0; 0, 0, d|| \quad \text{for } p \geq 3,
\]

where \((e + f \alpha, d) \in \mathbb{F}^*_p \times \mathbb{F}^*_p\) such that
\((-1, -1) \in \langle (e + f \alpha, d) \rangle\) and \(e + f \alpha\) is a fixed element of
order \(h\), where \(h \mid p^2 - 1\) but \(h \nmid p - 1\), and set \(n = [h, |d|]\).

\[
\mathcal{M}_{12}(p, h, d, i, j) = \mathcal{M}(G_{12}(p, h, d); x^i, t_{(1,0,1)}x^{\frac{in}{2}}),
\]

where \(i \in \mathbb{Z}^*_n/(\mathbb{Z}^*_n)^{2+}\), \(j = 0\) for \(p = 2\), and \(j = 1\) for \(p \geq 3\).
(1.3) $G_{13}(p, t_1, t_2, t_3) = T : \langle x \rangle,$

$$x = [t_1; t_2; t_3],$$

where $p \geq 5$, let $(t_1, t_2, t_3) \in \mathbb{Z}_p^* \times \mathbb{Z}_p^* \times \mathbb{Z}_p^*$ such that $(-1, -1, -1) \in \langle (t_1, t_2, t_3) \rangle$ and $t_1$, $t_2$ and $t_3$ are distinct integer, and set $n = [|t_1|, |t_2|, |t_3|] \geq 4.$

$M_{13}(p, t_1, t_2, t_3, i) = M(G_{13}(p, t_1, t_2, t_3); x^i, t_{(1,1,1)\times \frac{n}{2}}),$ 

where $i \in \mathbb{Z}_n^*/(\mathbb{Z}_n^*)^{2+}$ if $t_{i_1}^k = t_{i_1}, t_{i_2}^k = t_{i_3}$ and $t_{i_3}^k = t_{i_2}$ for some $k \in \mathbb{Z}_n^*$; $i \in \mathbb{Z}_n^*/(\mathbb{Z}_n^*)^{3+}$ if $t_{i_1}^k = t_{i_2}, t_{i_2}^k = t_{i_3}$ and $t_{i_3}^k = t_{i_1}$ for some $k \in \mathbb{Z}_n^*$; and $i \in \mathbb{Z}_n^*$ other cases, where \{i_1, i_2, i_3\} = \{1, 2, 3\}.
(2) \( G_2(p, t_1, t_2) = \langle a, b, x | a^{p^2} = b^p = x^n = 1, [a, b] = 1, a^x = a^{t_1}, b^x = b^{t_2} \rangle, \)

where \( p \geq 5, \) let \((t_1, t_2) \in \mathbb{Z}_{p^2}^* \times \mathbb{Z}_p^*\) such that \( |t_1| \mid (p - 1), (-1, -1) \in \langle (t_1, t_2) \rangle \) and \( t_1 \not\equiv t_2 (\text{mod } p); \) and set \( n = \lceil \lceil t_1 \rceil, \lceil t_2 \rceil \rceil \geq 4. \)

\( M_2(p, t_1, t_2, i) = M(G_2(p, t_1, t_2); x^i, abx^{n/2}), \)

where \( i \in \mathbb{Z}_n^*. \)
(3) \( G_3(p, n) = \langle a, x | a^{p^3} = x^n = 1, a^x = a^t \rangle, \)

where \( p \geq 3, \) \( n \) is an even divisor of \( p - 1 \) with \( n \geq 2, \) and let \( t \) be any fixed element of order \( n \) in \( \mathbb{Z}_{p^3}^*. \)

\[ \mathcal{M}_3(p, n, i) = \mathcal{M}(G_3(p, n); x^i, ax^{n^2/2}), \]

where \( i \in \mathbb{Z}_n^*. \)
Define two groups:

\[ G_{41}(p, t_1, t_2) = \langle a, b, x | a^p = b^p = c^p = x^n = 1, [a, b] = c, a^x = a^{t_1}, b^x = b^{t_2}, c^x = c^{t_1 t_2} \rangle, \]

where \( p \geq 5 \), let \((t_1, t_2) \in \mathbb{Z}_p^* \times \mathbb{Z}_p^*\) such that \((-1, -1) \in \langle (t_1, t_2) \rangle\) and \( t_1 \neq t_2 \), and set \( n = [|t_1|, |t_2|] \geq 4 \).

\[ M_{41}(p, t_1, t_2, i) = M(G_{41}(p, t_1, t_2); x^i, abc^{\frac{p-1}{2}} x^{\frac{n}{2}}), \]

where \( i \in \mathbb{Z}_n^*/(\mathbb{Z}_n^*)^{2+} \) if \( t_1^k = t_2 \) and \( t_2^k = t_1 \) for some \( k \in \mathbb{Z}_n^* \); \( i \in \mathbb{Z}_n^* \) for other cases.
\[(4.2) \quad G_{42}(p, n) = \langle a, b, x | a^p = b^p = c^p = x^n = 1, [a, b] = c, a^x = a^{e_1}b^{f_1}, b^x = a^{e_2}b^{f_2}, c^x = c \rangle, \]

\[(e_1, f_1, e_2, f_2) = (1, 1, 1, 0) \text{ for } p = 2 \text{ and } n = 3; \]

\[(e_1, f_1, e_2, f_2) = (e, f \theta, f, e) \text{ for } p \geq 3, \]

where \( n \mid p^2 - 1 \) but \( n \nmid p - 1 \), \( e + f \alpha \) is a fixed element of order \( n \) in \( \mathbb{F}_{p^2}^* \).

\[\mathcal{M}_{42}(p, n, i, j) = \mathcal{M}(G_{42}(p, n); x^i, ac \frac{jef \theta(1-e+f)}{4(e-1)} x^{jn} ), \]

where \( i = \pm 1 \) and \( j = 0 \) for \( p = 2 \), or \( i \in \mathbb{Z}_n^* \cap \{1, 2, \cdots, \frac{n}{2}\} \) and \( j = 1 \) for \( p \geq 3. \)
Suppose that $|P| = p^4$. Then $G$ and $M$ are given by

(1) $G_1(p, h) = \langle a, b, x | a^{p^3} = b^p = x^h = 1, a^b = a^{1+p^2}, a^x = a^e, b^x = b \rangle$,

where $p \geq 3$, $n = ph$ and $h$ any even divisor of $p - 1$, and let $e$ be any fixed element of order $h$ in $\mathbb{Z}_{p^2}^*$.

$M_1(p, h, i, j) = M(G_1(p, h); b^i x^i, a x^{\frac{h}{2}})$,

where $i \in \mathbb{Z}_p^*$ and $j \in \mathbb{Z}_h^*$. 
(2) \( G_2(p, h) = \langle a, b, x \mid a^{p^2} = b^p = c^p = x^h = 1, [a, b] = c, [c, a] = [b, c] = 1, a^x = a^e, b^x = b \rangle. \)

where \( p \geq 3, n = ph \) and \( h \geq 2 \) is an even divisor \( p - 1 \), and let \( e \) be any fixed element of order \( h \) in \( \mathbb{Z}_{p^2}^* \).

\[ M_2(p, h, i) = M(G_2(p, e); bx^i, acx^{h/2}), \]

where \( i \in \mathbb{Z}_h^* \).
(3) \( G_3(p, t_1, t_2) = \langle a, b, c, x | a^{p^2} = b^p = c^p = x^h = 1, a^b = a^{1+p}, [a, c] = [b, c] = 1, a^x = a^{t_1}, b^x = b, c^x = c^{t_2} \rangle, \)

where \( p \geq 5, \ n = ph \) and let \( h \mid p - 1, \) let \((t_1, t_2) \in \mathbb{Z}_{p^2}^* \times \mathbb{Z}_p^* \) such that \( |t_1| = h, \ t_1 \neq t_2 \) and \( \langle (t_1, t_2) \rangle \) contains \((-1, -1)\).

\( M_3(p, t_1, t_2, i, j) = M(G_3(p, t_1, t_2); b^i x^j, acx^h), \)

where \( i \in \mathbb{Z}_{p}^* \) and \( j \in \mathbb{Z}_h^*. \)
(4) $G_4(p, t_1, t_2) = \langle a, b, d, x | a^p = b^p = c^p = d^p = x^h = 1, [a, b] = c, [a, c] = [b, c] = [a, d] = [b, d] = 1, a^x = a, b^x = b^{t_1}, d^x = d^{t_2} \rangle$,

where $p \geq 5, \ n = ph$, let $(t_1, t_2) \in \mathbb{Z}_p^* \times \mathbb{Z}_p^*$ such that $t_1 = \theta^{p-1 \over h}, \ t_1 \neq t_2$ and $\langle (t_1, t_2) \rangle$ contains $(-1, -1)$, let $h = [|t_1|, |t_2|]$ with $h \geq 4$ is even.

$\mathcal{M}_4(p, t_1, t_2, i) = \mathcal{M}(G_4(p, t_1, t_2); ax^i, bdx^{h \over 2})$,

where $i \in \mathbb{Z}_h^*$. 
(5) \( G_5(p, h) = \langle a, b, x | a^{p^2} = b^p = c^p = x^h = 1, [a, b] = c, [a, c] = 1, [c, b] = a^{ip}, a^x = a^t, b^x = b \rangle, \)

where \( p \geq 3 \) and let \( t \) be any fixed element of order \( h \) in \( \mathbb{Z}_{p^2}^*. \)

\( M_5(p, h, j, k) = M(G_5(p, h); b^j x^k, a^{\frac{h}{2}}), \)

where \( j \in \mathbb{Z}_p^* \cap \{1, 2, \cdots, \frac{p-1}{2}\} \) and \( k \in \mathbb{Z}_h^*. \)
\begin{equation}
G_6(p, t_1, t_2, t_3) = \langle a, b, x | a^{p^2} = b^p = c^p = x^h = 1, [a, b] = c, [c, a] = a^p, [c, b] = 1, a^x = a^{t_1} c^{t_3}, b^x = b^{t_2} c^{\frac{1-t_2}{2}} \rangle,
\end{equation}

\[ t_1 \not\equiv t_2 \pmod{p}, (-1, -1) \in \langle (t_1, t_2) \rangle \text{ and } \frac{t_1^h - pht_3}{2} \equiv 1 \pmod{p^2}. \]

\[ M_6(p, t_1, t_2, t_3, i, j, k) = \]

\[ M(G_6(p, t_1, t_2, t_3); c^i x^j, ab^k c^{\frac{-k-t_3}{1-t_1}} x^{\frac{h}{2}}), \]

where \( i, k \in \mathbb{Z}_p^\star \) and \( j \in \mathbb{Z}_h^\star. \)
(7) Define three affine subgroups and the corresponding maps:

(7.1) \( G_{71}(p, t) = \langle a, b, x \mid a^p = b^p = c^p = d^p = x^h = 1, [a, b] = c, [c, a] = 1, [c, b] = d, a^x = a^t, b^x = b \rangle, \)

where \( p \geq 5 \) and let \( t \) be any fixed element of order \( h \) in \( \mathbb{Z}_p^*; \)

\( \mathcal{M}(p, t, i) = \mathcal{M}(G_{71}(p, t); bx^i, ax^{\frac{h}{2}}), \)

where \( i \in \mathbb{Z}_p^*. \)
\( G_{72}(p, t) = \langle a, b, x \mid a^p = b^p = c^p = d^p = x^h = 1, [a, b] = c, [c, a] = 1, [c, b] = d, a^x = a, b^x = b^t \rangle, \) 

where \( p \geq 5 \) and let \( t \) be any fixed element of order \( h \) in \( \mathbb{Z}_p^*; \) 

\( \mathcal{M}_{72}(p, t, i) = \mathcal{M}(G_{72}(p, t); a x^i, b x^{\frac{h}{2}}), \) 

where \( i \in \mathbb{Z}_h^*. \)
\[(7.3) \quad G_{73}(p, t_1, t_2) = \langle a, b, x \mid a^p = b^p = c^p = d^p = x^h = 1, [a, b] = c, [c, a] = 1, [c, b] = d, a^x = a^{t_1} c^{\frac{t_1 - 1}{2}}, b^x = b^{t_2} \rangle,\]

where \( p \geq 5 \) and let \((t_1, t_2) \in \mathbb{Z}_p^* \times \mathbb{Z}_p^* \) such that \( t_1 t_2 \equiv 1(\text{mod} \ p), \ t_1 \neq t_2 \) and \((-1, -1) \in \langle (t_1, t_2) \rangle;\)

\[\mathcal{M}_{73}(p, t_1, t_2, i, j) = \mathcal{M}(G_{73}(p, t_1, t_2), c^i x^j, a b x^{\frac{h}{2}}),\]

where \( i \in \mathbb{Z}_p^* \) and \( j \in \mathbb{Z}_h^*.\)
Define two subgroups and the corresponding maps:

(8.1) \[ G_{81}(2, 2) = \langle a, b \mid a^8 = b^2 = 1, a^b = a^{-1} \rangle \]

\[ M_{81}(2, 2) = M(G_{81}(2, 2), b, ab). \]

(8.2) \[ G_{82}(2, 2) = \langle a, b, c \mid a^4 = b^2 = c^2 = [a, c] = [b, c] = 1, a^b = a^{-1} \rangle \]

\[ M_{82}(2, 2, i) = M(G_{82}(2, 2), b, ab). \]
4. Further woks:

1. NORM of order $p^3$

2. Classify RM of order $p^3$ with multiple edges.
Thank You Very Much!