On hamiltonian cycles in Cayley graphs with commutator subgroup of order $pq$

Dave Witte Morris
University of Lethbridge, Alberta, Canada
http://people.uleth.ca/~dave.morris/talks.shtml
Dave.Morris@uleth.ca
Example

*Cayley graph*

\[\text{Cay}(\mathbb{Z}_m \oplus \mathbb{Z}_n; \{\pm (1, 0), \pm (0, 1)\})\]

vertices: elements of \(\mathbb{Z}_m \oplus \mathbb{Z}_n\)
edges: \(x \rightarrow x \pm (1, 0)\)
and \(x \rightarrow x \pm (0, 1)\)

has hamiltonian cycle

Defn. \(\text{Cay}(G; S)\) for group \(G\) and \(S \subseteq G\) with \(S = S^{-1}\)
vertices = elt’s of \(G\)  edge \(x \rightarrow xs\) for \(x \in G, s \in S\)

Exercise

\(G\) abelian \(\Rightarrow \forall S, \text{Cay}(G; S)\) has a hamiltonian cycle
(if connected, i.e., if \(\langle S \rangle = G\)).
Exercise

$G$ abelian $\Rightarrow \forall S, \text{Cay}(G; S)$ has a hamiltonian cycle
(if connected, i.e., if $\langle S \rangle = G$).

Open Problem (~1970)

¿Every connected Cayley graph has a hamiltonian cycle?

Many papers on this topic
[Marušič, Kutnar, Šparl, Alspach, Morris$^2$, Gallian, . . . ]

¿Can we find a ham cycle if $G$ is almost abelian?

Question: What is the next best thing to abelian?

Group theorist’s answer: nilpotent. [Moravec minicourse]

Remark. Open for nilpotent groups (but not $p$-groups).

(Cubic Cayley graphs on nilpotent groups have a ham path.)
¿ Can we find a ham cycle if $G$ is almost abelian?

Recall. commutator subgroup $G' = \langle ghg^{-1}h^{-1} \mid g, h \in G \rangle$.

$G$ abelian $\iff G' = \{e\} \iff |G'| = 1$.

¿ Can we find a ham cycle if $|G'|$ is small?

**Theorem** (Marušič, Durnberger, Keating-Witte 1985)

Cay$(G; S)$ has a ham cycle if $|G'| = p$ (prime).

Open problem. Find ham cycle if $G' = \mathbb{Z}_2 \times \mathbb{Z}_2$.

**Open Problem** (Marušič 1985)

Show Cay$(G; S)$ has ham cycle if $|G'| = p_1 p_2$. ($p_1 \neq p_2$)
Open Problem (Marušič 1985)

Show \( \text{Cay}(G; S) \) has ham cycle if \(|G'| = p_1 p_2\). \((p_1 \neq p_2)\)

Work in progress:

\(\checkmark\) \(G\) nilpotent \hspace{1cm} \(\checkmark\) \(|G|\) odd

[\(\text{Ghaderpour-Morris}\)] \hspace{1cm} [\(\text{Morris}\)]

\(\checkmark\) ? \(p_1 = 2\) \hspace{1cm} (in progress)

Hardest case: \(|G'|\) odd, but \(|G/G'|\) even (and small).

Proofs use voltage graphs. \hspace{1cm} [Ellingham minicourse]

\(G/G'\) is abelian, so \(\text{Cay}(G/G'; \overline{S})\) has ham cyc.

Lift this to a hamiltonian cycle in \(\text{Cay}(G; S)\).
Cay\((G; S)\) has a ham cycle if \(|G'| = p\)

**Idea of proof.** Ham cyc in Cay\((G/G'; \overline{S})\):

\[
\overline{x_0} s_1 \overline{x_1} s_2 \overline{x_2} s_3 \overline{x_3} s_4 \cdots s_n \overline{x_n} \quad (= \overline{x_0} = \overline{e}).
\]

Then \(\overline{x_i} = \overline{x_{i-1}} s_i\), so \(\overline{x_n} = s_1 s_2 \cdots s_n\).

Let \(\pi = s_1 s_2 \cdots s_n \in G'\). ("voltage") \(\pi^{p-1}\)

There are *many* ham cycs in \(G/G'\).

Find one with \(\pi \neq e\) ("Marušić’s Method")

so \(\langle \pi \rangle = G'\).

Ham cycle in \(G'\) lifts to a path in \(G\) from \(e\) to \(\pi\).

Repeated lifts extend this to a hamiltonian cycle in \(G\).
"Marušič’s Method"

Ham cycle in \( \text{Cay}(G/G'; \bar{S}) \):

Other ham cycles:

Voltage not all same (usually), so some voltage \( \neq e \).
