Skeletal Polyhedra, Polygonal Complexes, and Nets

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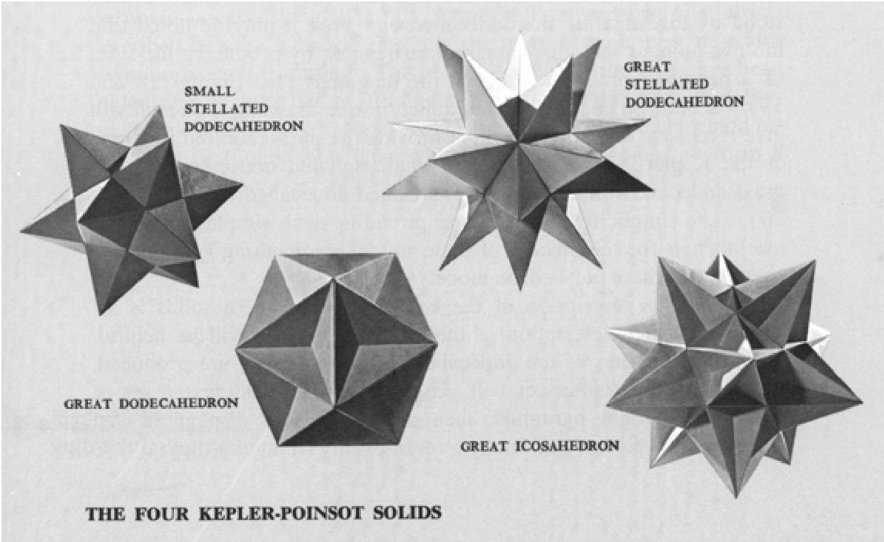
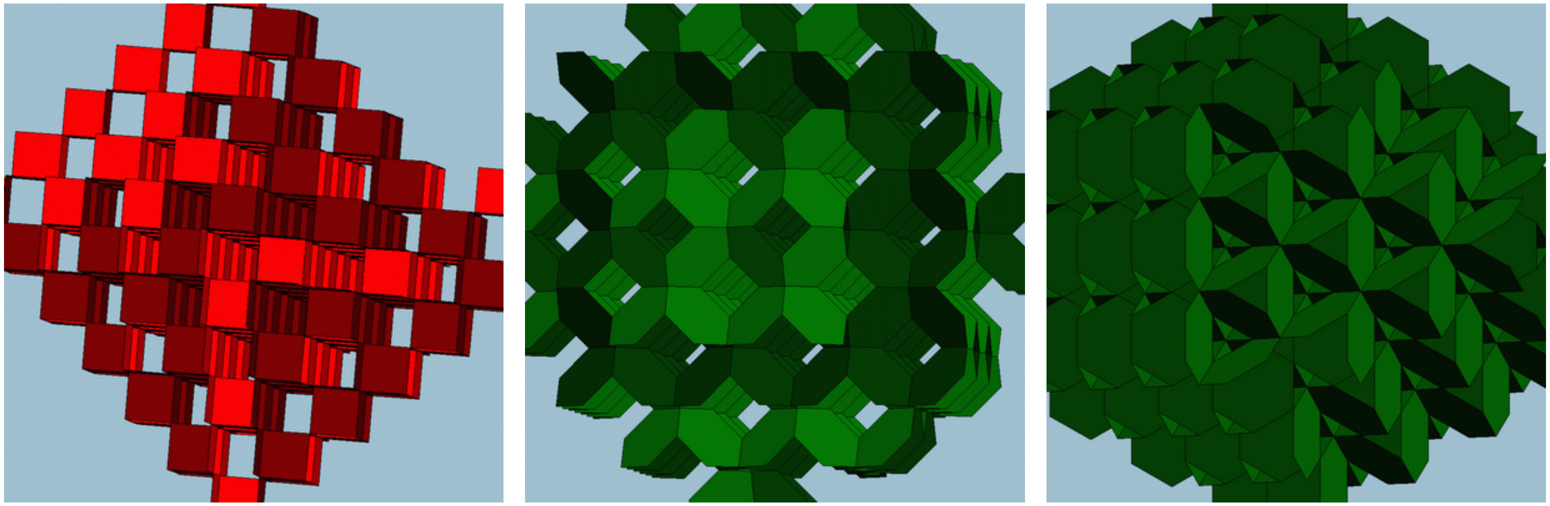
Polyhedra With the passage of time, many changes in point of view about polyhedral structures and their symmetry.

Platonic (solids, convexity), Kepler-Poinsot (star polygons),



Icosahedron: $\{3, 5\}$

Petrie-Coxeter polyhedra (convex faces, infinite), ...



Skeletal approach to polyhedra and symmetry!

- Branko Grünbaum (1970's) — geometrically and combinatorially.

- Allow skew faces! Restore the symmetry in the definition of "polyhedron"! Graph-theoretical approach!

- What are the regular polyhedra in ordinary space?
Answer: Grünbaum-Dress Polyhedra.

- The group theory forces *skew* faces and vertex-figures! General reflection groups.

Polyhedron

A polyhedron P in $\mathbb{E}^3$ is a (finite or infinite) family of simple polygons, called *faces*, such that

- each edge of a face is an edge of just one other face,
- all faces incident with a vertex form one circuit,
- P is connected,
- each compact set meets only finitely many faces (discreteness).

All traditional polyhedra are polyhedra in this generalized sense.

Highly symmetric polyhedra P

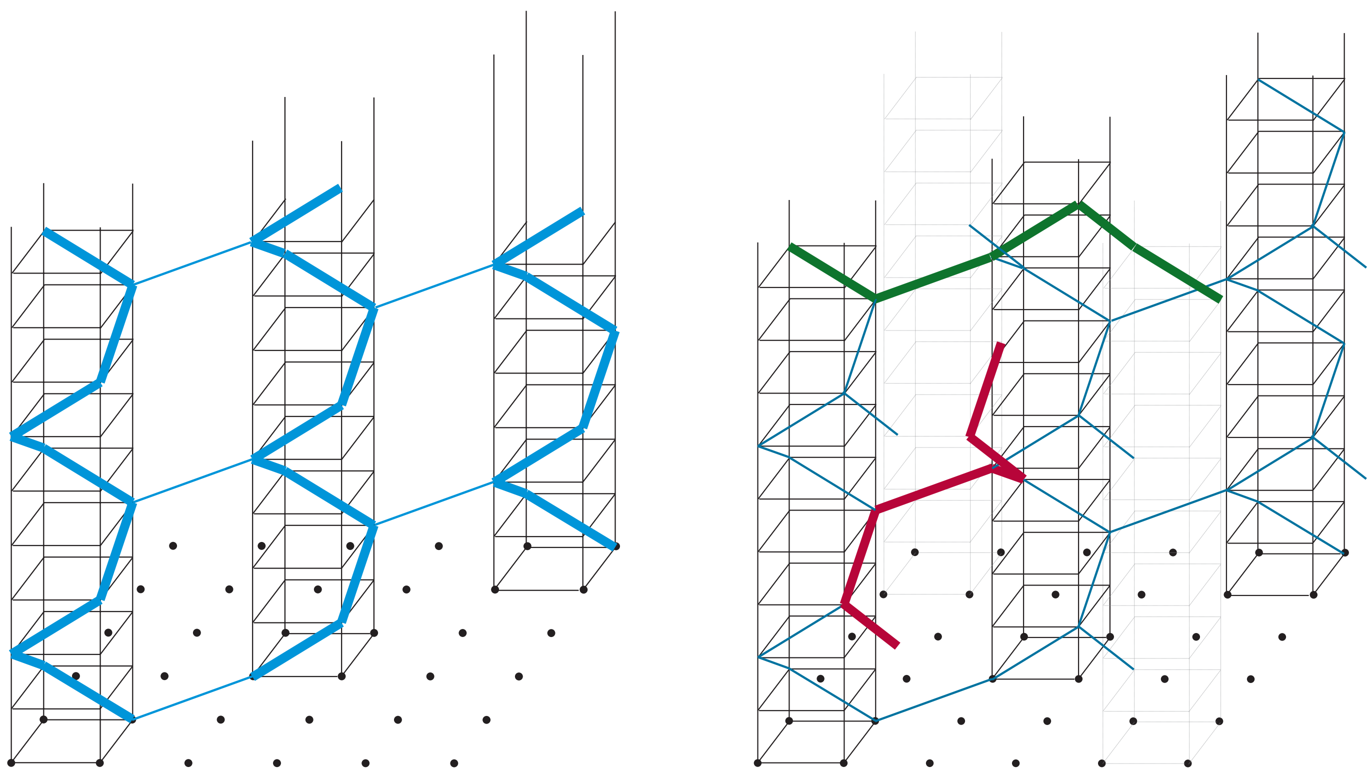
- Conditions on the geometric symmetry group $G(P)$ of P.

- P is regular if $G(P)$ is transitive on the flags.

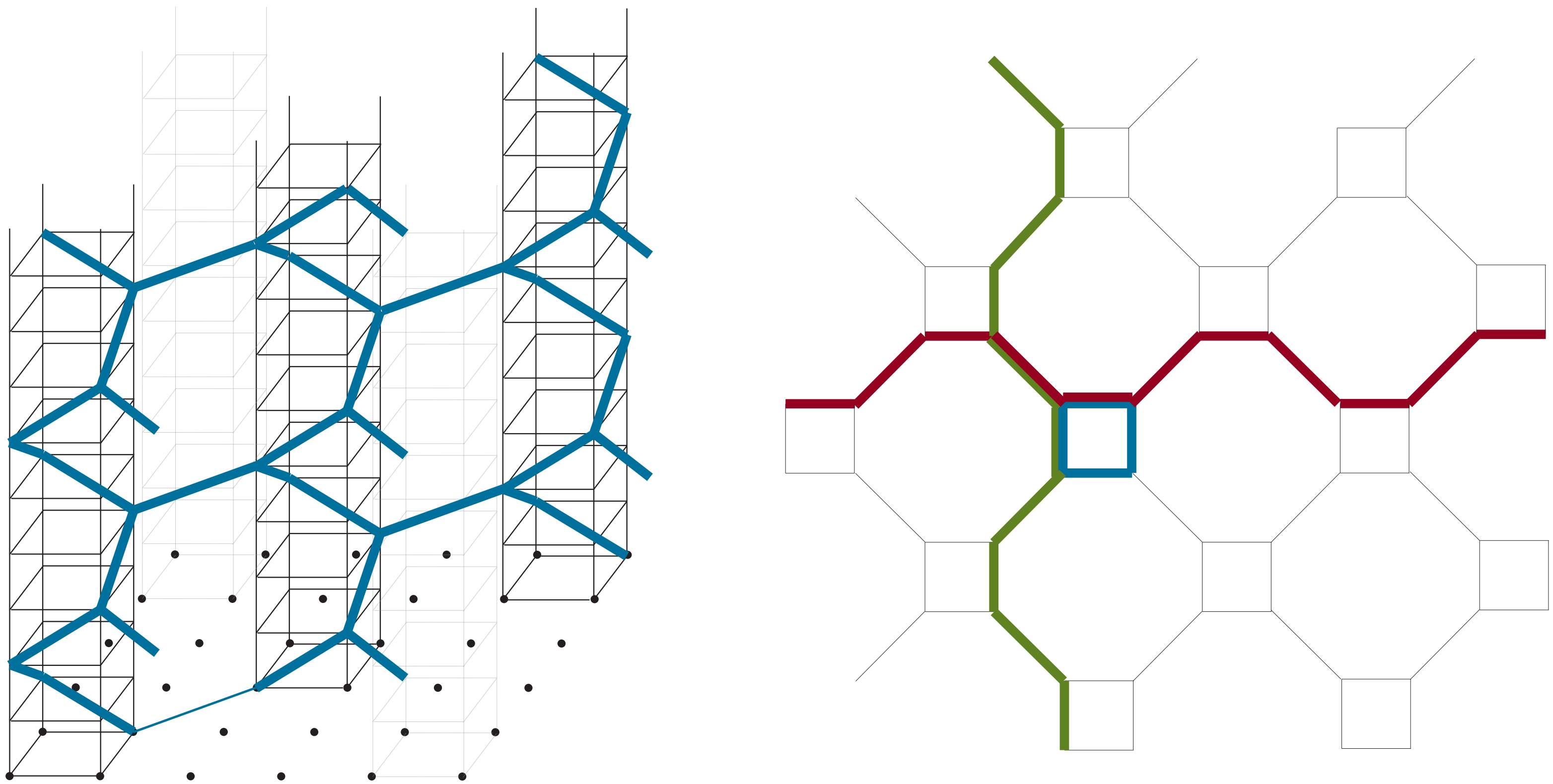
Flag: incident triple of a vertex, an edge, and a face.

- P is chiral if $G(P)$ has two orbits on the flags such that adjacent flags are in distinct orbits.

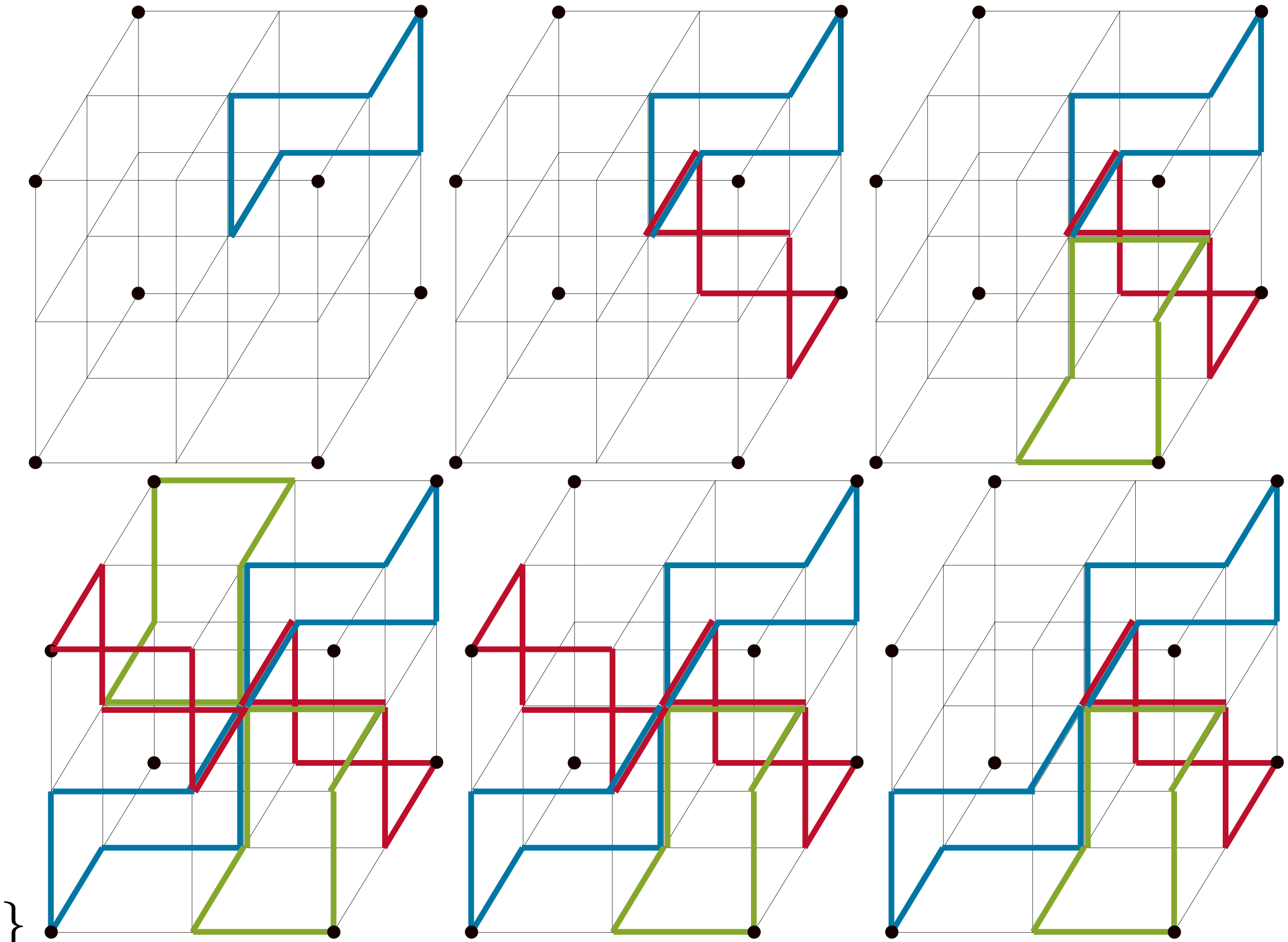
- Other interesting classes: Archimedean (regular faces, vertex-transitive); vertex-transitive (isogonal); face-transitive (isohedral); edge-transitive (isotoxal); ....



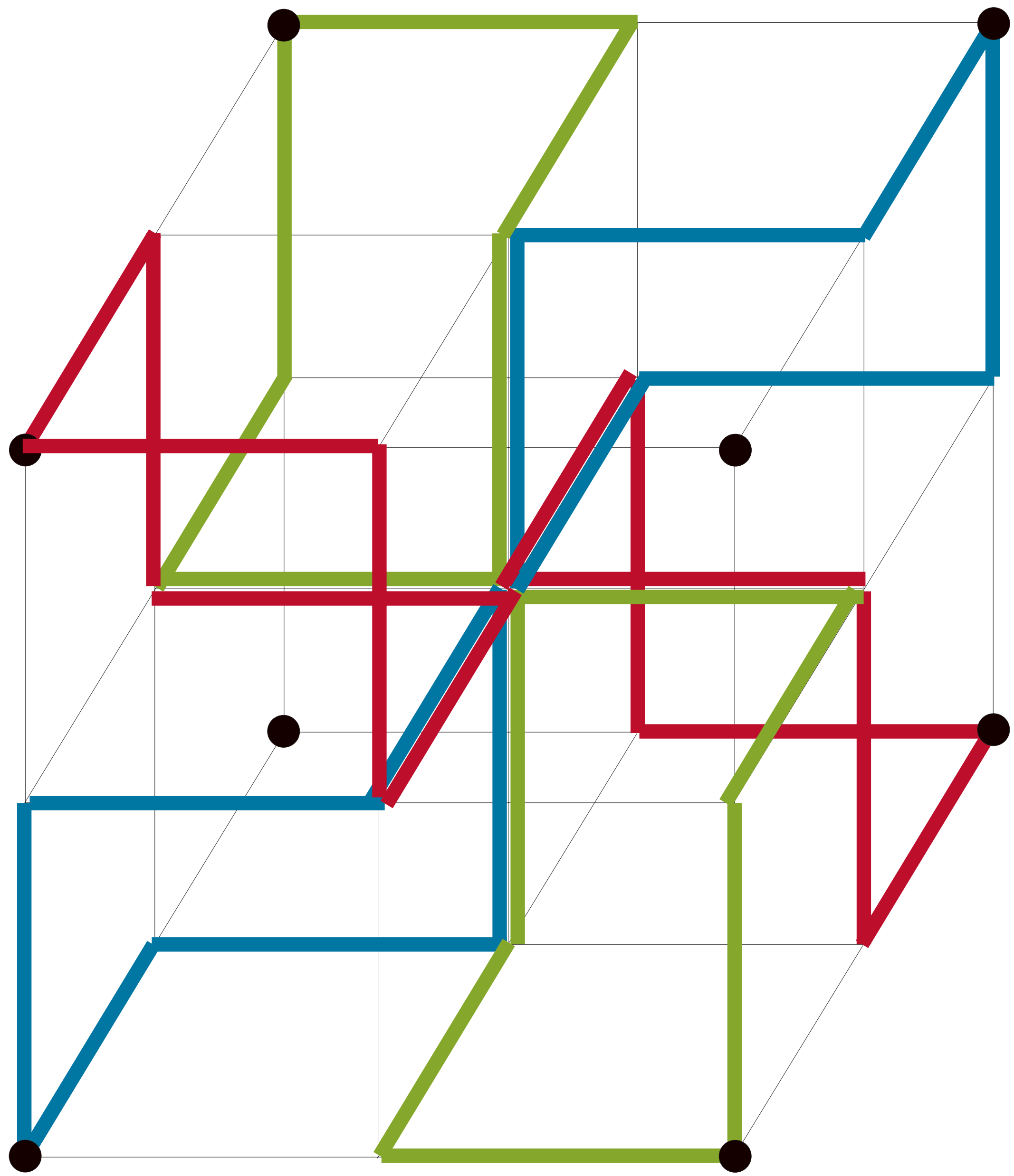
The helix-faced regular polyhedron $\{\infty, 3\}^{(b)}$*, with symmetry group requiring the single extra relation* $(R_0R_1)^4(R_0R_1R_2)^3 = (R_0R_1R_2)^3(R_0R_1)^4$.



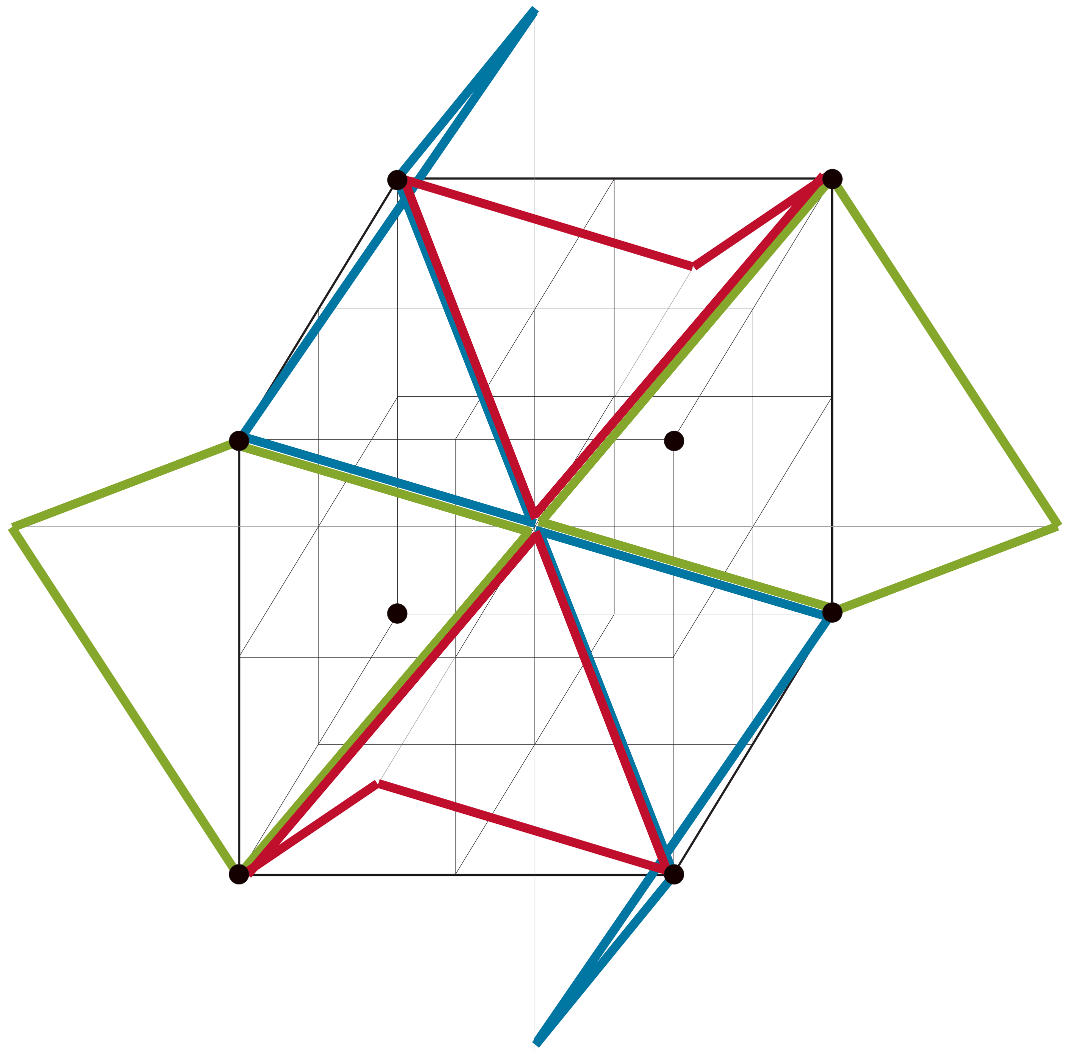
Helix-faced polyhedron $\{\infty, 3\}^{(b)}$



$\{6,6\}$



$\{6, 6\}$



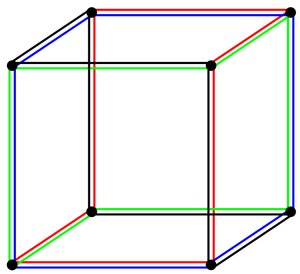
$\{4, 6\}$

Regular Polyhedra in $\mathbb{E}^3$

Grünbaum (70's), Dress (1981); McMullen & S. (1997)

18 finite polyhedra: 5 Platonic, 4 Kepler-Poinsot, 9 Petrials.

(2 full tetrahedral symmetry, 4 full octahedral, 12 full icosahedral)

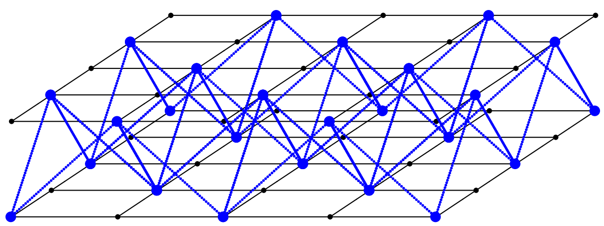


Petrie dual of the cube

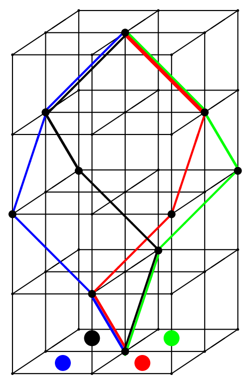
30 apeirohedra (infinite polyhedra). Crystallographic groups!

6 planar (3 regular tessellations and their Petrials)

12 reducible apeirohedra. Blends of a planar polyhedron and a linear polygon (segment or tessellation).



Square tessellation blended with the line segment. Symbol $\{4,4\}\#\{\,\}$



The square tessellation blended with a line tessellation. Each vertical column over a square is occupied by exactly one helical facet spiraling around the column. Symbol $\{4,4\}\#\{\infty\}$

12 irreducible apeirohedra.

$$\begin{array}{ccccccc}
\{\infty,4\}_{6,4} & \xleftrightarrow{\pi} & \{6,4|4\} & \xleftrightarrow{\delta} & \{4,6|4\} & \xleftrightarrow{\pi} & \{\infty,6\}_{4,4} \\
 & & \sigma\downarrow & & \downarrow\eta & & \\
 & & \{\infty,4\}_{\cdot,*3} & & \{6,6\}_4 & \xrightarrow{\varphi_2} & \{\infty,3\}^{(a)} \\
 & & & & \pi\updownarrow & & \updownarrow\pi \\
 & & \{6,4\}_6 & \xleftrightarrow{\delta} & \{4,6\}_6 & \xrightarrow{\varphi_2} & \{\infty,3\}^{(b)} \\
 & & \sigma\delta\downarrow & & \downarrow\eta & & \\
 & & \{\infty,6\}_{6,3} & \xleftrightarrow{\pi} & \{6,6|3\} & &
\end{array}$$

$\eta\colon R_0R_1R_0, R_2, R_1$; $\sigma = \pi\delta\eta\pi\delta\colon R_1, R_0R_2, (R_1R_2)^2$; $\varphi_2\colon R_0, R_1R_2R_1, R_2$

Breakdown by mirror vector (for R_0, R_1, R_2)

mirror vector	$\{3,3\}$	$\{3,4\}$	$\{4,3\}$	faces	vertex-figures
(2,1,2)	$\{6,6\|3\}$	$\{6,4\|4\}$	$\{4,6\|4\}$	planar	skew
(1,1,2)	$\{\infty,6\}_{4,4}$	$\{\infty,4\}_{6,4}$	$\{\infty,6\}_{6,3}$	helical	skew
(1,2,1)	$\{6,6\}_4$	$\{6,4\}_6$	$\{4,6\}_6$	skew	planar
(1,1,1)	$\{\infty,3\}^{(a)}$	$\{\infty,4\}_{\cdot,*3}$	$\{\infty,3\}^{(b)}$	helical	planar

Polyhedra in the last row occur in two enantiomorphic forms. Still, geometrically regular!

Presentations for the symmetry group are known. Fine Schläfli symbol signifies defining relations. Extra relations specify order of $R_0R_1R_2$, $R_0R_1R_2R_1$, or $R_0(R_1R_2)^2$.

How about regular polyhedra in higher dimensions?

- Coxeter's regular skew polyhedra in $\mathbb{E}^4$ (1930's) — convex faces and skew vertex-figures:

  $\{4,4|r\}\,(r \geq 3)$, $\{4,6|3\}$, $\{6,4|3\}$, $\{4,8|3\}$, $\{8,4|3\}$

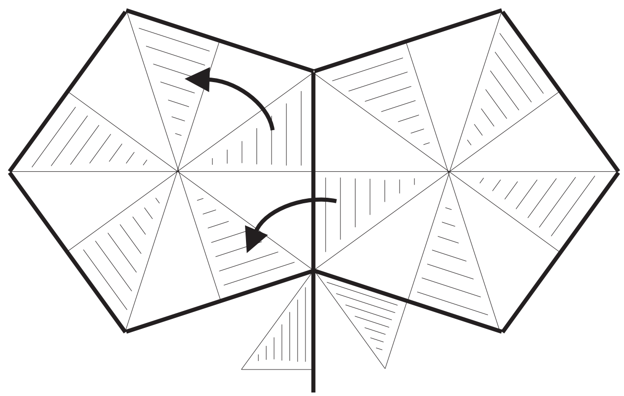
- Arocha, Bracho & Montejano (2000), Bracho (2000):
  regular polyhedra in $\mathbb{E}^4$ with *planar* faces and skew vertex-figures

- McMullen (2007):
  all regular polyhedra in $\mathbb{E}^4$

Chiral Polyhedra in $\mathbb{E}^3$

S., 2004, 2005

- Two flag orbits, with adjacent flags in different orbits.
- Local picture



- Maximal "rotational" symmetry but no "reflexive" symmetry! Irreflexible!

(Regularity: maximal "reflexive" symmetry.)

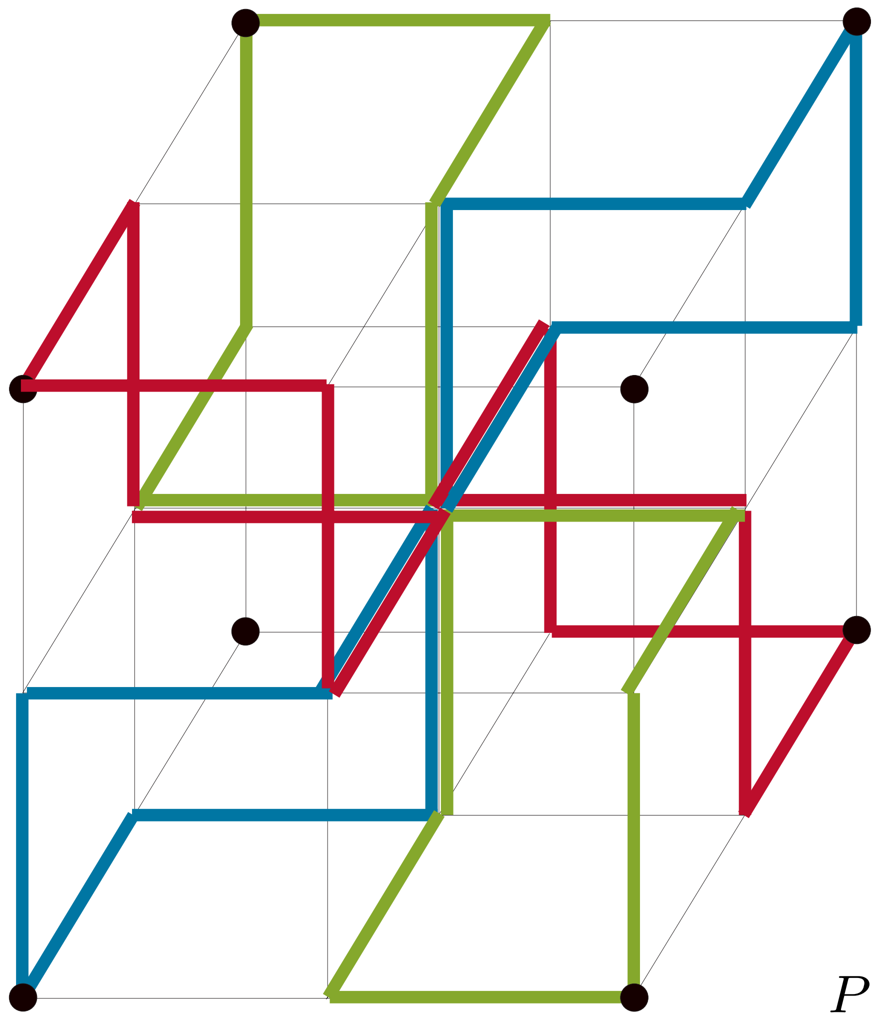
- No classical examples! No finite chiral polyhedra in $\mathbb{E}^3$.

Three Classes of Finite-Faced Chiral Polyhedra

(S_1, S_2 rotatory reflections, hence skew faces and skew vertex-figures.)

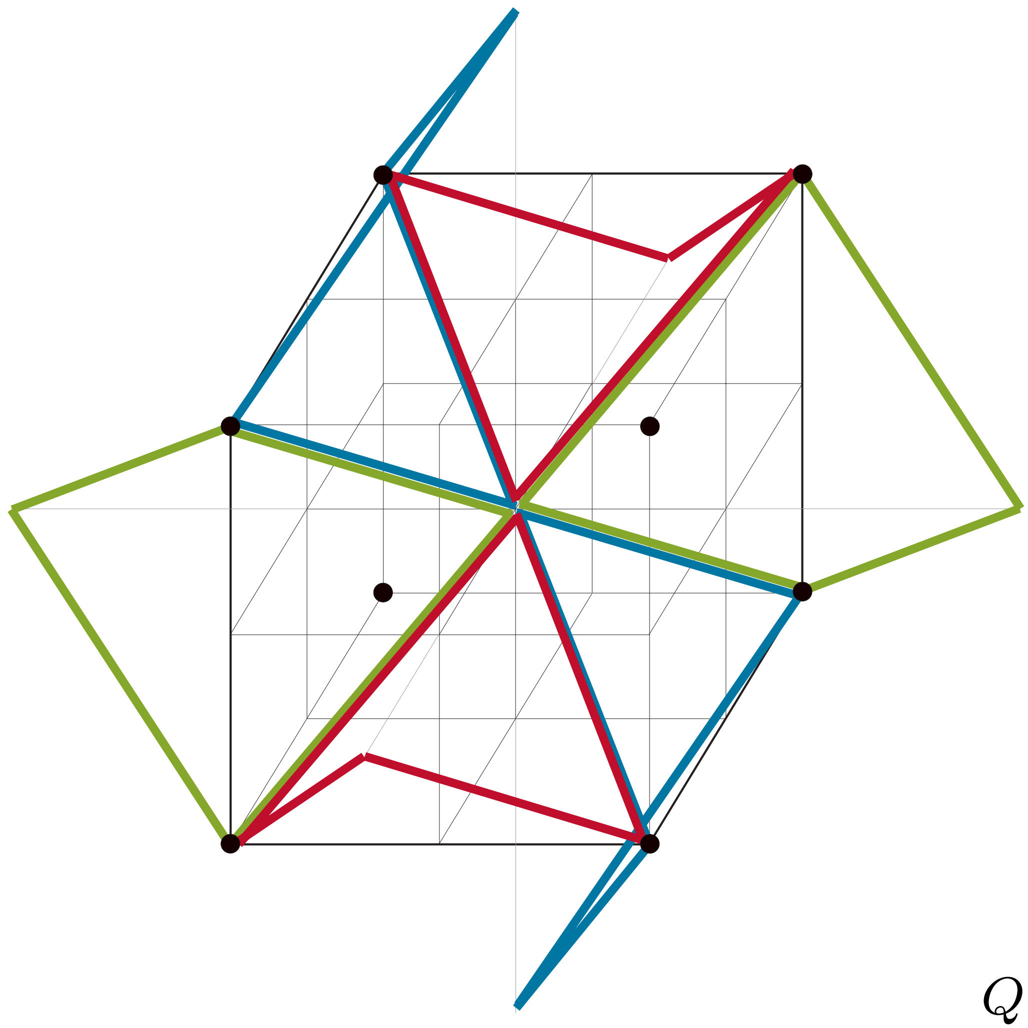
Schläfli	$\{6,6\}$	$\{4,6\}$	$\{6,4\}$
Notation	$P(a,b)$	$Q(c,d)$	$Q(c,d)^*$
Param.	$a,b \in \mathbb{Z}$, $(a,b)=1$	$c,d \in \mathbb{Z}$, $(c,d)=1$	$c,d \in \mathbb{Z}$, $(c,d)=1$
	geom. self-dual $P(a,b)^* \cong P(a,b)$		
Special gr	$[3,3]^+ \times \langle -I \rangle$	$[3,4]$	$[3,4]$
Regular cases	$P(a,-a)=\{6,6\}_4$ $P(a,a)=\{6,6\|3\}$	$Q(a,0)=\{4,6\}_6$ $Q(0,a)=\{4,6\|4\}$	$Q(a,0)^*=\{6,4\}_6$ $Q(0,a)^*=\{6,4\|4\}$

Vertex-sets and translation groups are known!



$P(1,0)$, of type $\{6,6\}$

Neighborhood of a single vertex.



$Q(1,1)$, of type $\{4,6\}$

Neighborhood of a single vertex.

Three Classes of Helix-Faced Chiral Polyhedra

(S_1 screw motion, S_2 rotation; helical faces and planar vertex-figures.)

Schläfli symbol	$\{\infty,3\}$	$\{\infty,3\}$	$\{\infty,4\}$
Helices over	triangles	squares	triangles
Special group	$[3,3]^+$	$[3,4]^+$	$[3,4]^+$
Relationships	$P(a,b)^{\varphi 2}$ $a \neq b$ (reals)	$Q(c,d)^{\varphi 2}$ $c \neq 0$ (reals)	$Q^*(c,d)^{\kappa}$ c,d reals
Regular cases	$\{\infty,3\}^{(a)}$ $= P(1,-1)^{\varphi 2}$ $= \{6,6\}_4^{\varphi 2}$	$\{\infty,3\}^{(b)}$ $= Q(1,0)^{\varphi 2}$ $= \{4,6\}_6^{\varphi 2}$	$\{\infty,4\}_{\cdot,*3}$ self- Petrie

Vertex-sets and translation groups are known!

Remarkable facts

- Essentially: any two finite-faced polyhedra of the same type are non-isomorphic.

$$P(a,b) \cong P(a',b') \text{ iff } (a',b') = \pm(a,b), \pm(b,a).$$

$$Q(c,d) \cong Q(c',d') \text{ iff } (c',d') = \pm(c,d), \pm(-c,d).$$

- Finite-faced polyhedra are intrinsically (combinatorially) chiral! [Pellicer & Weiss 2009]

- Helix-faced polyhedra combinatorially regular! Combin. only 3 polyhedra! Chiral helix-faced polyhedra are chiral deformations of regular helix-faced polyhedra! [P&W 2009]

- Chiral helix-faced polyhedra unravel Platonic solids! Coverings

$$\{\infty,3\} \mapsto \{3,3\}, \ \ \{\infty,3\} \mapsto \{4,3\}, \ \ \{\infty,4\} \mapsto \{3,4\}.$$

Polytopes of Higher Ranks

Regular: McMullen 2000's

In $\mathbb{R}^4$: 34 finite of rank 4; 14 infinite of rank 5.

Chiral: Bracho, Hubard & Pellicer (2014)

Examples of chiral polytopes of rank 4 in $\mathbb{E}^4$.

Regular Polygonal Complexes in $\mathbb{E}^3$ (with D.Pellicer)

(Hybrids of polyhedra and incidence geometries. Polyhedral geometries.)

A polygonal complex K in $\mathbb{E}^3$ is a family of simple polygons, called *faces*, such that

- each edge of a face is an edge of exactly r faces ($r \geq 2$);
- the vertex-figure at each vertex is a connected graph, possibly with multiple edges;
- the edge graph of K is connected;
- each compact set meets only finitely many faces (discreteness).

K is regular if its geometric symmetry group $G(K)$ is transitive on the flags of K.

..... The End .....

Thank you

Abstract

Skeletal polyhedra and polygonal complexes in 3-space are finite, or infinite periodic, geometric edge graphs equipped with additional polyhedra-like structure determined by faces (simply closed planar or skew polygons, zig-zag polygons, or helical polygons). The edge graphs of the infinite polyhedra and complexes are periodic nets. We discuss classification results for skeletal polyhedra and polygonal complexes in 3-space by distinguished transitivity properties of the symmetry group, as well as the relevance of these structures for the classification of crystal nets.