Colorings of affine and projective spaces

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SYGN IV, July 4, 2014, Rogla

UNIVERZA NA PRIMORSKEM
UNIVERSITÀ DEL LITORALE

REPUBLIKA SLOVENIJA
MINISTRSTVO ZA IZOBRAŽEVANJE, ZNANOST IN ŠPORT

Naložba v vašo prihodnost
OPERACIJO DELNO FINANCIRA EVROPSKA UNIJA
Evropski socialni sklad

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Hypergraph coloring

A *C-hypergraph* $\mathcal{H} = (X, \mathcal{C})$ has an underlying vertex set X and a set system $\mathcal{C}$ over X. A *vertex coloring* of $\mathcal{H}$ is a mapping ϕ from X to a set of colors $\{1, 2, \ldots, k\}$.
A *rainbow-free k-coloring* is a mapping $\phi : X \to \{1, \ldots, k\}$ such that each $\mathcal{C}$*-edge* $C \in \mathcal{C}$ has at least two vertices with the *common* color.
The *upper chromatic number* of $\mathcal{H}$, denoted by $\bar{\chi}(\mathcal{H})$, is the largest k admitting a rainbow-free k-coloring.

Coloring of projective spaces

Let Π be an n-dimensional projective space and $0 < d < n$ be an integer. Then Π may be considered as a hypergraph, whose vertices and hyperedges are the points and the d-dimensional subspaces of the space, respectively.

Upper chromatic number of finite planes

Theorem (Bacsó, Tuza (2007))

1. *As $q \to \infty$, any projective plane Π_q of order q satisfies*

$$\bar{\chi}(\Pi_q) \leq q^2 - q - \sqrt{q}/2 + o(\sqrt{q}).$$

2. *If q is a square, then the Galois plane of order q satisfies*

$$\bar{\chi}(\mathrm{PG}(2,q)) \geq q^2 - q - 2\sqrt{q}.$$

Best known result

Theorem (Bacsó, Héger, Szőnyi (2012))

Let $q = p^h$, p prime. Let $\tau_2 = 2(q+1) + c$ denote the size of the smallest double blocking set in $\mathrm{PG}(2,q)$. Suppose that one of the following two conditions holds:

1. $206 \le c \le c_0 q - 13$, *where* $0 < c_0 < 2/3$,
   $q \ge q(c_0) = 2(c_0+2)/(2/3 - c_0) - 1$, *and*
   $p \ge p(c_0) = 50c_0 + 24$.
2. $q > 256$ *is a square.*

Then $\mathrm{dec}(\mathrm{PG}(2,q)) = \tau_2 - 1$, and equality is reached if and only if the only color class having more than one point is a smallest double blocking set.

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Then $\mathrm{dec}(\mathrm{PG}(2,q)) = \tau_2 - 1$, and equality is reached if and only if the only color class having more than one point is a smallest double blocking set.

For arbitrary finite projective planes this result may be false or hopeless to prove.

Balanced colorings

Usually a rainbow–free coloring has a lot of color classes with one element each, and one big color class.

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Let $\phi : X \to \{1, \dots, k\}$ be a rainbow-free k-coloring. If $X_i = \phi^{-1}(i)$. then $X_1 \cup \cdots \cup X_k = X$ is called *color partition* . The coloring ϕ is called *balanced,* if

$$-1 \leq |X_i| - |X_j| \leq 1$$

holds for all $i, j \in \{1, 2, \dots, k\}$.
The *balanced upper chromatic number* of $\mathcal{H}$, denoted by $\bar{\chi}_b(\mathcal{H})$, is the largest k admitting a balanced rainbow-free k-coloring.

Balanced colorings

Π_q a projective plane of order q, $v = q^2 + q + 1$.

Theorem

All balanced rainbow-free colorings of any projective plane of order q satisfies that each color class contains at least three points. Thus

$$\overline{\chi}_b(\Pi_q) \leq \frac{q^2 + q + 1}{3}.$$

Example in the case $q = 3$.
The projective plane of order 3 have $3^2 + 3 + 1 = 13$ points and 13 lines. Take the vertices of a regular 13-gon $P_1P_2 \dots P_{13}$. The chords obtained by joining distinct vertices of the polygon have 6 $(= 3(3 + 1)/2)$ different lengths. Choose 4 $(= 3 + 1)$ vertices of the regular 13–gon so that all the chords obtained by joining pairs of these points have different lengths. Four vertices define $4 \times 3/2 = 6$ chords. The vertices P_1, P_2, P_5 and P_7 form a good subpolygon, Λ_0.

The cyclic model

The points of the plane are the vertices of the regular 13-gon.

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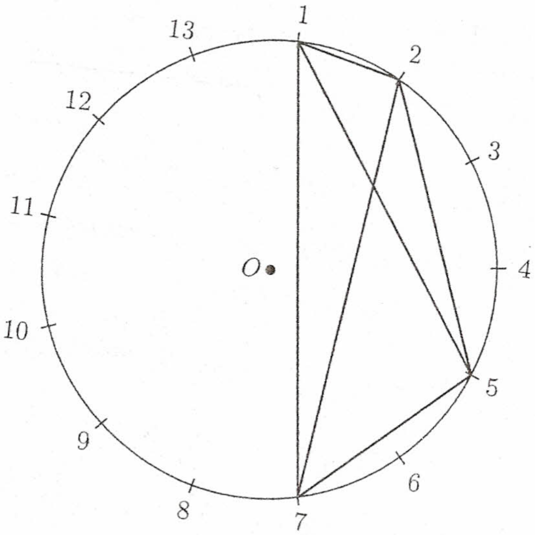
The incidence is the set theoretical inclusion.

Cyclic model



The cyclic model

We can construct a projective plane of order q, if we are able to choose $q + 1$ vertices of the regular $(q^2 + q + 1)$-gon in such a way that no two chords spanned by the choosen vertices have the same length.

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One can find easily such sets of vertices if q is a prime power (algebraic method, points of $\mathrm{PG}(2,q) \leftrightarrow$ elements of the cyclic group $\mathrm{GF}^\star(q^3)/\mathrm{GF}^\star(q)$.

Each known cyclic plane has prime power order.

Balanced colorings

Theorem

Let Π_q be a cyclic projective plane of order q and let p be the smallest nontrivial divisor of $v = q^2 + q + 1$. Then Π_q has a balanced rainbow–free coloring with $\frac{v}{p}$ color classes. Thus

$$\overline{\chi}_b(\Pi_q) \geq \frac{q^2 + q + 1}{p}.$$

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Define the color classes as:

$$C_i = \{i, i + \frac{v}{p}, ..., i + (p-1)\frac{v}{p}\}.$$

Each line contains a pair of points of the form $\{j, j + \frac{v}{p}\}$.

Balanced colorings

Corollary

If $q \equiv 1 \pmod 3$ then each cyclic plane of order q has a balanced rainbow–free coloring with $\frac{v}{3}$ color classes. Therefore, in this case

$$\overline{\chi}_b(\Pi_q) = \frac{q^2+q+1}{3}.$$

Balanced colorings

Proposition

If $\mathbb{Z}_v$ *has a difference set* D *containing the subset* $\{0, 1, 3\}$, *then the corresponding cyclic plane of order* q *has a balanced rainbow–free coloring with* $\lfloor \frac{v}{3} \rfloor$ *color classes. Hence, in this case*

$$\overline{\chi}_b(\Pi_q) = \left\lfloor \frac{q^2 + q + 1}{3} \right\rfloor .$$

Balanced colorings

Theorem

Each cyclic projective plane of order q has a balanced rainbow-free coloring with at least $\frac{v}{6}$ color classes. Thus

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Corollary

Let Π_q be a cyclic projective plane of order q. Then

$$\frac{q^2+q+1}{6} \leq \overline{\chi}_b(\Pi_q) \leq \frac{q^2+q+1}{3}.$$

A *k-spread* of $\mathrm{PG}(n, q)$ is a set of pairwise disjoint k-dimensional subspaces which gives a partition of the points of the geometry.

Theorem

There exists a k-spread in $\mathrm{PG}(n, q)$ if and only if $(k+1)|(n+1)$.

Spreads in $\mathrm{PG}(n, q)$

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Theorem

There exists a k-spread in $\mathrm{PG}(n, q)$ if and only if $(k+1)|(n+1)$.

Proposition

Let $\mathcal{S}$ be an $(n-1)/2$-spread in $\mathrm{PG}(n, q)$. Then each hyperplane of $\mathrm{PG}(n, q)$ contains exactly one element of $\mathcal{S}$.

Let $H(q, n, d)$ be the hypergraph whose vertex-set is the set of points of $\mathrm{PG}(n, q)$ and the edges are the d-dimensional subspaces of $\mathrm{PG}(n, q)$.

Balanced colorings in $\mathrm{PG}(n, q)$

Let $H(q, n, d)$ be the hypergraph whose vertex-set is the set of points of $\mathrm{PG}(n, q)$ and the edges are the d-dimensional subspaces of $\mathrm{PG}(n, q)$.

Theorem

Let $n \geq 3$ be an odd number. Then $H(q, n, n-1)$ has a balanced rainbow-free coloring with $\frac{q^{n+1}-1}{q-1} - q^{\frac{n+1}{2}} - 1$ color classes. Thus

$$\overline{\chi}_b(H(q, n, n-1)) \geq \frac{q^{n+1}-1}{q-1} - q^{\frac{n+1}{2}} - 1.$$

Balanced colorings in $\mathrm{PG}(3, q)$

Theorem

Each balanced rainbow-free coloring of $H(q, 3, 2)$ has at most $q^3 + q$ color classes. Hence

$$\overline{\chi}_b(H(q, 3, 2)) = q^3 + q.$$

Theorem

If $H(q, n, d)$ has a balanced rainbow-free coloring with the additional property that each color class has the same size, say k), then $H(q, n+1, d)$ also has a balanced rainbow-free coloring with $(q^{n+1} - 1)/k(q-1)$ color classes.

Balanced colorings in $\mathrm{PG}(n, q)$

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Corollary

Let p be the smallest nontrivial divisor of $v = q^2+q+1$. Then $H(q,3,1)$ has a balanced rainbow-free coloring with $\frac{v}{p}$ color classes. In particular if $q \equiv 1 \pmod 3$ then

$$\overline{\chi}_b(H(q,3,1)) \geq \frac{q^2+q+1}{3}.$$

Balanced colorings in $\mathrm{PG}(3, q)$

Theorem

The size of the larger color classes in a balanced rainbow-free coloring of the points with respect to the lines in $\mathrm{PG}(3, q)$ is at least $2q + 2$, hence

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Corollary

If $q \equiv 1 \pmod 3$ then

$$\frac{q^2 + q + 1}{3} \leq \overline{\chi}_b(H(q, 3, 1)) \leq \frac{q^2 + 1}{2}.$$

END OF PART I

John Cleese

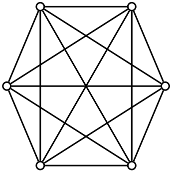
Graph decomposition

Definition

A decomposition of a simple graph $G = (V(G), E(G))$ *is a pair* $[G, \mathcal{D}]$ *where* $\mathcal{D}$ *is a set of induced subgraphs of* G*, such that every edge of* G *belongs to exactly one subgraph in* $\mathcal{D}$*.*

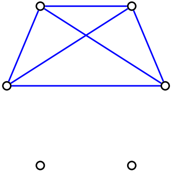
Definition

A decomposition of a simple graph $G = (V(G), E(G))$ is a pair $[G, \mathcal{D}]$ where $\mathcal{D}$ is a set of induced subgraphs of G, such that every edge of G belongs to exactly one subgraph in $\mathcal{D}$.



K_6

Graph decomposition



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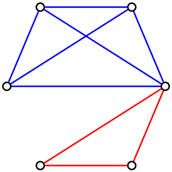

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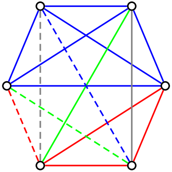

K_6

Coloring

Definition

A coloring of a decomposition $[G, \mathcal{D}]$ with k colors is a surjective function that assigns to edges of G a color from a k-set of colors, such that all edges of $H \in \mathcal{D}$ have the same color. A coloring of $[G, \mathcal{D}]$ with k colors is proper, if for all $H_1, H_2 \in \mathcal{D}$ with $H_1 \neq H_2$ and $V(H_1) \cap V(H_2) \neq \emptyset$, then $E(H_1)$ and $E(H_2)$ have different colors.

Definition

The chromatic index $\chi'([G, \mathcal{D}])$ of a decomposition is the smallest number k for which there exists a proper coloring of $[G, \mathcal{D}]$ with k colors.

Coloring

Definition

A coloring of $[G, \mathcal{D}]$ *with* k *colors is complete if each pair of colors appears on at least a vertex of* G*. The pseudoachromatic index* $\psi'([G, \mathcal{D}])$ *of a decomposition is the largest number* k *for which there exist a complete coloring with* k *colors.*

Definition

The achromatic index $\alpha'([G, \mathcal{D}])$ *of a decomposition is the largest number* k *for which there exist a proper and complete coloring with* k *colors.*

Coloring

If $\mathcal{D} = E(G)$ then $\chi'([G, E])$, $\alpha'([G, E])$ and $\psi'([G, E])$ are the usual *chromatic*, *achromatic* and *pseudoachromatic indices* of G, respectively.

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Clearly we have that

$$\chi'([G,\mathcal{D}]) \leq \alpha'([G,\mathcal{D}]) \leq \psi'([G,\mathcal{D}]).$$

Conjecture (Erdős-Faber-Lovász)

For any decomposition $\mathcal{D}$ of K_v, given by complete graphs, satisfies the inequality

$$\chi'([K_v, \mathcal{D}]) \leq v.$$

Designs define decompositions of the corresponding complete graphs in the natural way. Identify the points of a (v, κ)-design $D = (\mathcal{V}, \mathcal{B})$ with the set of vertices of the complete graph K_v. Then the set of points of each block of D induces in K_v a subgraph isomorphic to K_κ and these subgraphs give a decomposition of K_v.

Known results

$\mathrm{PG}(n, q)$ can be regarded as a $(\frac{q^{n+1}-1}{q-1}, q+1)$–design, where the set of blocks are the set of lines of $\mathrm{PG}(n, q)$.

Theorem (Beutelspacher, A. – Jungnickel, D. – Vanstone, S.A.)

If $\mathcal{D}$ is the n-dimensional finite projective space, then

$$\chi'(\mathcal{D}) \leq v,$$

the EFL Conjecture is true for finite projective spaces.

Projective planes

Let Π_q be any finite projective plane of order q. Then $v = q^2 + q + 1$ is the number of points in Π_q. It is not hard to see that

$$\chi'(\Pi_q) = \alpha'(\Pi_q) = \psi'(\Pi_q) = v.$$

Achromatic index

Theorem

$$\alpha'(\mathrm{PG}(5,q)) \geq c\frac{v^{1.4}}{\kappa - 1}, \text{ where } v = \frac{q^6-1}{q-1}, \text{ and } c \text{ a fixed constant}$$

Theorem

Let $\mathcal{D}$ be a (v, κ)–design. Then

$$\psi'(\mathcal{D}) \leq \frac{\sqrt{v}(v-1)}{\kappa - 1} < \frac{v^{1.5}}{\kappa - 1}.$$

Theorem

Let A_q be any affine plane of order q. Then

$$\psi'(\mathrm{A}_q) = \left\lfloor \frac{(q+1)^2}{2} \right\rfloor .$$

3-dimensional affine space

Theorem

Let $\mathrm{AG}(3,q)$ *be the 3-dimensional affine space of order* q*. Then*

- $\frac{(q^2+q)(q+1)+2}{2} \leq \alpha'(\mathrm{AG}(3,q)) \leq \left\lfloor (q^3+q^2+q)\sqrt{q} - \frac{1}{2}q^3 \right\rfloor,$
- $q^3+1 \leq \psi'(\mathrm{AG}(3,q)) \leq \left\lfloor (q^3+q^2+q)\sqrt{q} - \frac{1}{2}q^3 \right\rfloor.$

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- $q^3+1 \le \psi'(\mathrm{AG}(3,q)) \le \left\lfloor (q^3+q^2+q)\sqrt{q} - \frac{1}{2}q^3 \right\rfloor$.

Upper estimate: a refinement of the General Upper Bound Theorem.

Lower estimates: clever constructions.

Theorem

Let $\mathrm{AG}(4, q)$ *be the 4-dimensional affine space of order* q. *Then*

$$\frac{q^5 + q^4 + q^3 + q}{2} \leq \alpha'(\mathrm{AG}(4, q)) \leq \left\lfloor \frac{q^6 - q^2}{q - 1} \right\rfloor .$$

4-dimensional affine space

Theorem

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$$\frac{q^5 + q^4 + q^3 + q}{2} \leq \alpha'(\mathrm{AG}(4, q)) \leq \left\lfloor \frac{q^6 - q^2}{q - 1} \right\rfloor .$$

Theorem

Let $\mathrm{AG}(4, q)$ *be the 4-dimensional affine space of order* q. *Then*

$$\frac{q^5 + q^4 + q^3 + q^2}{2} \leq \psi'(\mathrm{AG}(4, q)) \leq \left\lfloor \frac{q^6 - q^2}{q - 1} \right\rfloor .$$

Work in progress

If the dimension is even:

Theorem

- $\frac{q^{3k-1}}{2} < \alpha'(\mathrm{AG}(2k, q)) < \frac{q^{3k}-q^k}{q-1}$,
- $\frac{q^{3k-1}}{2} < \psi'(\mathrm{AG}(2k, q)) < \frac{q^{3k}-q^k}{q-1}$.

Work in progress

If the dimension is odd:

Theorem

- $\frac{q^{3k}}{2} < \alpha'(\mathrm{AG}(2k+1,q)) < q^{3k}\sqrt{q}$,
- $q^{3k} < \psi'(\mathrm{AG}(2k+1,q)) < q^{3k}\sqrt{q}$.

THANKS FOR YOUR ATTENTION!