Pentavalent symmetric graphs of order twice a prime power

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Outline

Definitions

- All graphs mentioned in this talk are **simple, connected and undirected**, unless otherwise stated.
- An **automorphism** of a graph $\Gamma = (V, E)$ is a permutation on the vertex set V preserving the adjacency.
- All automorphisms of a graph $\Gamma = (V, E)$ forms the **automorphism group** of Γ, denoted by $\mathrm{Aut}(\Gamma)$.
- An **s-arc** in a graph Γ is an ordered $(s+1)$-tuple $(v_0, v_1, \cdots, v_{s-1}, v_s)$ of vertices of Γ such that v_{i-1} is adjacent to v_i for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$.

Transitivity of graphs

Let Γ is a connected graph, and let $G \leq \mathrm{Aut}(\Gamma)$ be a subgroup of $\mathrm{Aut}(\Gamma)$.

- Γ is **(G, s)-arc-transitive** or **(G, s)-regular** if G acts transitively or regularly on s-arcs.
- A (G, s)-arc-transitive graph is **(G, s)-transitive** if G acts transitively on s-arcs but not on $(s + 1)$-arcs.
- A graph Γ is said to be **s-arc-transitive**, **s-regular** or **s-transitive** if it is $(\mathrm{Aut}(\Gamma), s)$-arc-transitive, $(\mathrm{Aut}(\Gamma), s)$-regular or $(\mathrm{Aut}(\Gamma), s)$-transitive.
- 0-arc-transitive means **vertex-transitive**, and 1-arc-transitive means **arc-transitive** or **symmetric**.

Normal cover

Let Γ be a symmetric graph, and let $N \trianglelefteq \mathrm{Aut}(\Gamma)$ be a normal subgroup of $\mathrm{Aut}(\Gamma)$.

- The **quotient graph** Γ_N of Γ relative to N is defined as the graph with vertices the orbits of N on $V(\Gamma)$ and with two orbits adjacent if there is an edge in Γ between those two orbits.
- If Γ and Γ_N have the same valency, Γ is a **normal cover** (also **regular cover**) of Γ_N, and Γ_N is a **normal quotient** of Γ.
- A graph Γ is called **basic** if Γ has no proper normal quotient.
- Γ_N is simple, but the covering theory works for non-simple graph when we take the quotient by a semiregular subgroup: **an arc of Γ_N corresponds to an orbits of arcs under the semiregular subgroup**, which produces **multiedges, semiedges, loops**.

Research plan for symmetric graph

- There are often two steps to study a symmetric graph Γ:
  (1) **Investigating quotient graph Γ_N for some normal subgroup N of $\mathrm{Aut}(\Gamma)$**;
  (2) **Reconstructing the original graph Γ from the normal quotient Γ_N by using covering techniques**.
- It is usually done by taking N as large as possible, and then the graph Γ is reduced a ‘basic graph’.
- This idea was first introduced by Praeger [27, 28, 29] for **locally primitive graphs**.

Basic graphs

- A **locally primitive graph** is a vertex-transitive graph with a vertex stabilizer acting primitively on its neighbors.
- A locally primitive graph Γ is basic $\Leftrightarrow$ **every nontrivial normal subgroup of $\mathrm{Aut}(\Gamma)$ has one or two orbits**.
- A graph Γ is **quasiprimitive** if every nontrivial normal subgroup of $\mathrm{Aut}(\Gamma)$ is transitive, and is **biquasiprimitive** if $\mathrm{Aut}(\Gamma)$ has a nontrivial normal subgroup with two orbits but no such subgroup with more than two orbits.
- For locally primitive graphs, **basic graphs are equivalent to quasiprimitive or biquasiprimitive graphs**.

Basic graphs

Some known results about basic graphs.

- Baddeley [2] gave a detailed description of 2-arc-transitive quasiprimitive graphs of twisted wreath type.
- Ivanov and Praeger [13] completed the classification of 2-arc-transitive quasiprimitive graphs of affine type.
- Li [15, 16, 17] classified quasiprimitive 2-arc-transitive graphs of odd order and prime power order.
- Symmetric graphs of diameter 2 admitting an affine-type quasiprimitive group were investigated by Amarra et al [1].
- ....

Cubic symmetric basic graphs of order $2p^n$

D.Ž. Djoković and G.L. Miller [6, Propositions 2-5]

Let Γ be a cubic (G, s)-transitive graph for some group $G \leq \mathrm{Aut}(\Gamma)$ and integer $s \geq 1$, and let $v \in V(X)$. Then $s \leq 5$ and $G_v \cong \mathbb{Z}_3$, S_3, $\mathrm{S}_3 \times \mathbb{Z}_2$, S_4 or $\mathrm{S}_4 \times \mathbb{Z}_2$ for $s = 1, 2, 3, 4$ or 5, respectively.

- Y.-Q. Feng and J.H. Kwak in [Cubic symmetric graphs of order twice an odd prime-power, J. Aust. Math. Soc. 81 (2006), 153-164] determined all the cubic symmetric basic graphs of order $2p^n$.
- In 2012, Devillers et al [5] constructed an infinite family of biquasiprimitive 2-arc transitive cubic graphs.

Tetravalent symmetric basic graphs of order $2p^n$

Potočnik [26], for partial results also see [19, 18, 15]

Let Γ be a connected (G, s)-transitive tetravalent graph, and let v be a vertex in Γ. Then

(1) $s = 1$, G_v is a 2-group;
(2) $s = 2$, $G_v \cong \mathrm{A}_4$ or S_4;
(3) $s = 3$, $G_v \cong \mathrm{A}_4 \times \mathbb{Z}_3$, $(\mathrm{A}_4 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$ with $\mathrm{A}_4 \rtimes \mathbb{Z}_2 = \mathrm{S}_4$ and $\mathbb{Z}_3 \rtimes \mathbb{Z}_2 = \mathrm{S}_3$, or $\mathrm{S}_4 \times \mathrm{S}_3$;
(4) $s = 4$, $G_v \cong \mathbb{Z}_3^2 \rtimes \mathrm{GL}(2,3) = \mathrm{AGL}(2,3)$;
(5) $s = 7$, $G_v \cong [3^5] \rtimes \mathrm{GL}(2,3)$.

- J.-X. Zhou and Y.-Q. Feng in [Tetravalent s-transitive graphs of order twice a prime power, J. Aust. Math. Soc. 88 (2010) 277-288] classified all the tetravalent symmetric basic graphs of order $2p^n$.

Pentavalent symmetric basic graphs of order $2p^n$

S.-T. Guo and Y.-Q. Feng [11, Theorem 1.1]

Let Γ be a connected pentavalent (G, s)-transitive graph for some group $G \leq \mathrm{Aut}(\Gamma)$ and integer $s \geq 1$, and let $v \in V(\Gamma)$. Then

(1) $s = 1$, $G_v \cong \mathbb{Z}_5$, D_5 or D_{10};
(2) $s = 2$, $G_v \cong F_{20}$, $F_{20} \times \mathbb{Z}_2$ A_5 or S_5;
(3) $s = 3$, $G_v \cong F_{20} \times \mathbb{Z}_4$, $\mathrm{A}_4 \times \mathrm{A}_5$, $\mathrm{S}_4 \times \mathrm{S}_5$, or $(\mathrm{A}_4 \times \mathrm{A}_5) \rtimes \mathbb{Z}_2$ with $\mathrm{A}_4 \rtimes \mathbb{Z}_2 \cong \mathrm{S}_4$ and $\mathrm{A}_5 \rtimes \mathbb{Z}_2 \cong \mathrm{S}_5$;
(4) $s = 4$, $G_v \cong \mathrm{ASL}(2, 4)$, $\mathrm{AGL}(2, 4)$, $\mathrm{A\Sigma L}(2, 4)$ or $\mathrm{A\Gamma L}(2, 4)$;
(5) $s = 5$, $G_v \cong \mathbb{Z}_2^6 \rtimes \Gamma\mathrm{L}(2, 4)$.

Problem

Determining pentavalent symmetric basic graphs of order $2p^n$.

Pentavalent symmetric basic graphs of order $2p^n$

Main Theorem

Each basic graph of connected pentavalent symmetric graphs of order $2p^n$ is isomorphic to one graph in the following table.

Γ	$\mathrm{Aut}(\Gamma)$	p	Normal Cayley graph
K_6	S_6	$p = 3$	No
FQ_4	$\mathbb{Z}_2^4 \rtimes \mathrm{S}_5$	$p = 2$	Yes
$\mathcal{CD}_p$	$\mathrm{S}_5 \,\mathrm{wr}\, \mathbb{Z}_2$	$p = 5$	No
	$\mathrm{PGL}(2, 11)$	$p = 11$	No
	$D_p \rtimes \mathbb{Z}_5$	$5 \mid (p-1)$	Yes
$\mathcal{CGD}_{p^3}$	$\mathrm{Dih}(\mathbb{Z}_p^3) \rtimes \mathbb{Z}_5$	$p = 5$	Yes
$\mathcal{CGD}_{p^2}^{[2]}$	$\mathrm{Dih}(\mathbb{Z}_p^2) \rtimes D_5$	$5 \mid (p \pm 1)$	Yes
$\mathcal{CGD}_{p^4}$	$\mathrm{Dih}(\mathbb{Z}_p^4) \rtimes \mathrm{S}_5$	$p \neq 2$ or 5	Yes

Pentavalent symmetric graphs of order $2p^n$

Reduction Theorem

Let p be a prime and let Γ be a connected pentavalent symmetric graph of order $2p^n$ with $n \geq 1$. Then Γ is a normal cover of one graph in the following table.

Γ	$\mathrm{Aut}(\Gamma)$	p	Normal Cayley graph
K_6	S_6	$p = 3$	No
FQ_4	$\mathbb{Z}_2^4 \rtimes \mathrm{S}_5$	$p = 2$	Yes
$\mathcal{CD}_p$	$\mathrm{S}_5 \,\mathrm{wr}\, \mathbb{Z}_2$	$p = 5$	No
	$\mathrm{PGL}(2, 11)$	$p = 11$	No
	$D_p \rtimes \mathbb{Z}_5$	$5 \mid (p-1)$	Yes
$\mathcal{CGD}^{[1]}_{p^2}$	$(\mathrm{Dih}(\mathbb{Z}_5^2) \rtimes F_{20})\mathbb{Z}_4$	$p = 5$	No
	$\mathrm{Dih}(\mathbb{Z}_p^2) \rtimes \mathbb{Z}_5$	$5 \mid (p-1)$	Yes
$\mathcal{CGD}^{[2]}_{p^2}$	$\mathrm{Dih}(\mathbb{Z}_p^2) \rtimes D_5$	$5 \mid (p \pm 1)$	Yes
$\mathcal{CGD}_{p^3}$	$\mathrm{Dih}(\mathbb{Z}_p^3) \rtimes \mathbb{Z}_5$	$5 \mid (p-1)$	Yes
$\mathcal{CGD}_{p^4}$	$\mathrm{Dih}(\mathbb{Z}_p^4) \rtimes \mathrm{S}_5$		Yes

Graph Constructions

- Let $D_p = \langle a, b \mid a^p = b^2 = 1, b^{-1}ab = a^{-1} \rangle$ be the dihedral group of order $2p$. For $p = 5$, let $\ell = 1$ and for $5 \mid (p-1)$, let ℓ be an element of order 5 in $\mathbb{Z}_p^*$.

$$\mathcal{CD}_p = \mathrm{Cay}(D_p, \{b, ab, a^{\ell+1}b, a^{\ell^2+\ell+1}b, a^{\ell^3+\ell^2+\ell+1}b\}) \qquad (1)$$

$\mathrm{Aut}(\mathcal{CD}_p)$ **was given by Cheng and Oxley [4].**

- Let $\mathrm{Dih}(\mathbb{Z}_p^2) = \langle a, d, h \mid a^p = d^p = h^2 = [a, d] = 1, h^{-1}ah = a^{-1}, h^{-1}dh = d^{-1} \rangle$. For $p = 5$, let $\ell = 1$, and for $5 \mid (p-1)$, let ℓ be an element of order 5 in $\mathbb{Z}_p^*$. Define

$$\mathcal{CGD}_{p^2}^{[1]} = \mathrm{Cay}(\mathrm{Dih}(\mathbb{Z}_p^2), \{h, ah, a^{\ell(\ell+1)^{-1}}d^{\ell^{-1}}h, a^{\ell}d^{(\ell+1)^{-1}}h, dh\}). \qquad (2)$$

- For $5 \mid (p \pm 1)$, let λ be an element in $\mathbb{Z}_p^*$ such that $\lambda^2 = 5$. Define

$$\mathcal{CGD}_{p^2}^{[2]} = \mathrm{Cay}(\mathrm{Dih}(\mathbb{Z}_p^2), \{h, ah, a^{2^{-1}(1+\lambda)}dh, ad^{2^{-1}(1+\lambda)}h, dh\}). \qquad (3)$$

Graph Constructions

- Let $\mathrm{Dih}(\mathbb{Z}_p^3) = \langle a, b, d, h \mid a^p = b^p = d^p = h^2 = [a, b] = [a, d] = [b, d] = 1, h^{-1}ah = a^{-1}, h^{-1}bh = b^{-1}, h^{-1}dh = d^{-1}\rangle$. For $p = 5$, let $\ell = 1$, and for $5 \mid (p-1)$, let ℓ be an element of order 5 in $\mathbb{Z}_p^*$. Define

$$\mathcal{CGD}_{p^3} = \mathrm{Cay}(\mathrm{Dih}(\mathbb{Z}_p^3), \{h, ah, bh, a^{-\ell^2} b^{-\ell} d^{-\ell^{-1}} h, dh\}). \quad (4)$$

- Let $\mathrm{Dih}(\mathbb{Z}_p^4) = \langle a, b, c, d, h \mid a^p = b^p = c^p = d^p = h^2 = [a, b] = [a, c] = [a, d] = [b, c] = [b, d] = [c, d] = 1, h^{-1}ah = a^{-1}, h^{-1}bh = b^{-1}, h^{-1}ch = c^{-1}, h^{-1}dh = d^{-1}\rangle$. Define

$$\mathcal{CGD}_{p^4} = \mathrm{Cay}(\mathrm{Dih}(\mathbb{Z}_p^4), \{h, ah, bh, ch, dh\}). \quad (5)$$

Automorphisms from Cayley graphs

Theorem

Let $\Gamma = \mathrm{Cay}(G, S)$ be one of the graphs defined in Eqs (2)-(5). Let P be a Sylow p-subgroup of $R(G)$ and let $A = \mathrm{Aut}(\Gamma)$. Then Γ is $N_A(P)$-arc-transitive.

(1) *Let* $\Gamma = \mathcal{CGD}_{p^2}^{[1]}$ $(p = 5$ *or* $5 \mid (p-1))$. *If* $p = 5$ *then* $N_A(R(\mathrm{Dih}(\mathbb{Z}_5^2)) \cong R(\mathrm{Dih}(\mathbb{Z}_5^2)) \rtimes F_{20}$ *and if* $5 \mid (p-1)$ *then* $N_A(R(\mathrm{Dih}(\mathbb{Z}_p^2))) \cong R(\mathrm{Dih}(\mathbb{Z}_p^2)) \rtimes \mathbb{Z}_5$. *Furthermore,* $|N_A(P)| \neq 20p^2$.

(2) *Let* $\Gamma = \mathcal{CGD}_{p^2}^{[2]}$ $(5 \mid (p \pm 1))$. *Then* $N_A(R(\mathrm{Dih}(\mathbb{Z}_p^2))) \cong R(\mathrm{Dih}(\mathbb{Z}_p^2)) \rtimes D_{10}$ *and* $|N_A(P)|$ *has a divisor* $20p^2$.

(3) *Let* $\Gamma = \mathcal{CGD}_{p^3}$ $(p = 5$ *or* $5 \mid (p-1))$. *If* $p = 5$ *then* $N_A(R(\mathrm{Dih}(\mathbb{Z}_5^3))) \cong R(\mathrm{Dih}(\mathbb{Z}_5^3)) \rtimes S_5$ *and if* $5 \mid (p-1)$ *then* $R(\mathrm{Dih}(\mathbb{Z}_p^3)) \rtimes \mathbb{Z}_5 \leq N_A(R(\mathrm{Dih}(\mathbb{Z}_p^3)))$.

(4) *Let* $\Gamma = \mathcal{CGD}_{p^4}$. *Then* $N_A(R(\mathrm{Dih}(\mathbb{Z}_p^4))) \cong R(\mathrm{Dih}(\mathbb{Z}_p^4)) \rtimes S_5$.

Let $A = \mathrm{Aut}(\mathrm{Cay}(G, S))$. Then $N_A(R(G)) = R(G) \rtimes \mathrm{Aut}(G, S)$ by Godsil [12], where $\mathrm{Aut}(G, S) = \{\alpha \in \mathrm{Aut}(G) \mid S^\alpha = S\}$.

Idea for the proof of Reduction Theorem

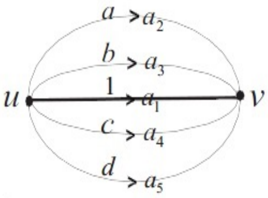
Key Lemma

Let p be a prime, and let Γ be a connected pentavalent G-arc-transitive graph of order $2p^n$ with $n \geq 2$, where $G \leq \mathrm{Aut}(\Gamma)$. Then every minimal normal subgroup of G is an elementary abelian p-group.

- Let M be a maximal normal subgroup of $A = \mathrm{Aut}(\Gamma)$ which has more than two orbits on $V(\Gamma)$. Then by Lorimer's result [22], Γ_M is a connected pentavalent A/M-arc-transitive graph of order $2p^m$ for some integer $m \leq n$. In particular, $\Gamma_M \cong K_6$ **or** $\mathcal{CD}_p$ **by Cheng and Oxley [4] if** $m = 1$.
- **Assume that** $m \geq 2$. Let N be a minimal normal subgroup of A/M. Then by Key Lemma, N is an elementary abelian p-group and by the maximality of M, N has at most two orbits on $V(\Gamma_M)$.

Idea for the proof of Reduction Theorem

- N **is transitive**: As N is abelian, N is regular, that is, Γ_M is a connected symmetric Cayley graph on the elementary abelian p-group N. Hence, $p = 2$ and $m \leq 4$. For $m = 4$, $N \cong \mathbb{Z}_2^5$ and $\Gamma_M \cong Q_5 \cong \mathcal{CGD}_{2^4}$. For $m \leq 3$, by McKay [25], $\Gamma_M \cong FQ_4$.
- N **has two orbits on** $V(\Gamma_M)$: Γ_M is a bipartite graph with the two orbits of N as its partite sets and N **acts regularly on each partite set**. In this case, **Γ_M is an elementary abelian symmetric covers of the Dipole Dip_5**.



Elementary abelian covers are Cayley graphs

Lemma

Let Γ be a bipartite graph and H an abelian semiregular automorphism group of Γ with the two partite sets of Γ as its orbits. Then Γ is a Cayley graph on $\mathrm{Dih}(H)$.

Idea for the proof:

$\Gamma = B_1 \cup B_2$, $B_1 = \{h \mid h \in H\}$ and $B_2 = \{h' \mid h \in H\}$.

For any $h, g \in H$, $h^g = hg$ and $(h')^g = (hg)'$.

Define α: $h \mapsto (h^{-1})'$, $h' \mapsto h^{-1}$, $h \in H$. Then $\alpha \in Aut(\Gamma)$ and $o(\alpha) = 2$. In particular, $\alpha g \alpha = g^{-1}$ and $\langle H, \alpha \rangle \cong \mathrm{Dih}(H)$.

Γ is a Cayley graph on $\mathrm{Dih}(H)$.

Idea for the proof of Reduction Theorem

To prove the Reduction Theorem, we need:

- **Prove the Key Lemma**.
- **Classify elementary abelian symmetric covers of Dip_5**.
- **Determine isomorphic problems between these covers**.
- **Determine full automorphism groups of these covers**.

The first is proved by group theory analysis

The last three are proved by covering theory, together with the automorphism information from Cayley graphs.

Idea for proof of the Key Lemma

Key Lemma

Let p be a prime, and let Γ be a connected pentavalent G-arc-transitive graph of order $2p^n$ with $n \geq 2$, where $G \leq \mathrm{Aut}(\Gamma)$. Then every minimal normal subgroup of G is an elementary abelian p-group.

Idea for the proof:

- By Guo and Feng's result about vertex stabilizer, $|G_v| \mid 2^9 \cdot 3^2 \cdot 5$ and thus $|G| \mid 2^{10} \cdot 3^2 \cdot 5 \cdot p^n$.
- Let N be a minimal normal subgroup of G. It suffices to prove that N is solvable.

Idea for the proof of the Key Lemma

- Suppose N is insolvable. Then $N = T^s$ for some integer $s \geq 1$, where T is a non-abelian simple $\{2,3,5,p\}$-group. As $|N| \mid 2^{10} \cdot 3^2 \cdot 5 \cdot p^n$, we have $N \cong \mathrm{A}_5$ with $p = 2$, A_6 with $p \in \{2,3\}$ or $\mathrm{PSU}(4,2)$ with $p = 3$.

- Suppose $N \cong \mathrm{A}_5$ or A_6. Then $|V(\Gamma)| = 8, 16$ or 18 and by McKay [25], $\Gamma \cong FQ_4$ which is impossible because $\mathrm{Aut}(\Gamma) \cong \mathbb{Z}_2^4 \rtimes \mathrm{S}_5$.

- Suppose $N \cong \mathrm{PSU}(4,2)$. Then $|V(\Gamma)| = 2 \cdot 3^3$ or $2 \cdot 3^4$ and $|N_v| = 2^6 \cdot 3 \cdot 5$, $2^6 \cdot 5$, $2^5 \cdot 3 \cdot 5$ or $2^5 \cdot 5$. Since $N_v \trianglelefteq G_v$ and G_v is known, $N_v \cong \mathrm{ASL}(2,4)$, which is impossible by MAGMA.

Elementary abelian symmetric covers of the Dipole Dip_5

Theorem

Let p be a prime and $\mathbb{Z}_p^n$ an elementary abelian group with $n \geq 2$. Let Γ be a connected symmetric $\mathbb{Z}_p^n$-cover of the dipole Dip_5. Then $2 \leq n \leq 4$ and

(1) *For $n = 2$, $\Gamma \cong \mathcal{CGD}_{p^2}^1$ ($p = 5$ or $5 \mid (p-1)$) or $\mathcal{CGD}_{p^2}^2$ ($5 \mid (p \pm 1)$), which are unique for a given order; $\mathrm{Aut}(\mathcal{CGD}_{5^2}^1) = (R(GD_{5^2}) \rtimes F_{20})\mathbb{Z}_4 \cong \mathbb{Z}_5 \cdot ((F_{20} \times F_{20}) \rtimes \mathbb{Z}_2)$ with $N_A(R(GD_{5^2})) = R(GD_{5^2}) \rtimes F_{20}$, $\mathrm{Aut}(\mathcal{CGD}_{p^2}^1) = R(GD_{p^2}) \rtimes \mathbb{Z}_5$ for $5 \mid (p-1)$, and $\mathrm{Aut}(\mathcal{CGD}_{p^2}^2) = R(GD_{p^2}) \rtimes D_{10}$;*

(2) *For $n = 3$, $\Gamma \cong \mathcal{CGD}_{p^3}$ ($p = 5$ or $5 \mid (p-1)$), which are unique for a given order; $\mathrm{Aut}(\mathcal{CGD}_{5^3}) = R(GD_{5^3}) \rtimes S_5$ and $\mathrm{Aut}(\mathcal{CGD}_{p^3}) = R(GD_{p^3}) \rtimes \mathbb{Z}_5$ for $5 \mid (p-1)$;*

(3) *For $n = 4$, $\Gamma \cong \mathcal{CGD}_{p^4}$ and $\mathrm{Aut}(\mathcal{CGD}_{p^4}) = R(GD_{p^4}) \rtimes S_5$.*

Elementary abelian symmetric covers of the Dipole Dip_5

Together with others, the full automorphism groups are computed by the following result.

Malnič[23, Theorem 4.2]

Let $\mathcal{P} : \widetilde{\Gamma} = Cov(\Gamma; \zeta) \mapsto \Gamma$ be a regular N-covering projection. Then an automorphism α of Γ lifts if and only if $\overline{\alpha}$ extends to an automorphism of N.

Together with others, the isomorphic problems are solved by the following result.

Malnič, Marušič, Potočnik [24, Corollary 3.3(a)]

Let $\mathcal{P}_1 : Cov(\Gamma; \zeta_1) \mapsto \Gamma$ and $\mathcal{P}_2 : Cov(\Gamma; \zeta_2) \mapsto \Gamma$ be two regular N-covering projections of a graph Γ. Then $\mathcal{P}_1$ and $\mathcal{P}_2$ are isomorphic if and only if there is an automorphism $\delta \in \mathrm{Aut}(\Gamma)$ and an automorphism $\eta \in \mathrm{Aut}(N)$ such that $(\zeta_1(W))^\eta = \zeta_2(W^\delta)$ for all fundamental closed walks W at some base vertex of Γ.

Classifications of pentavalent symmetric graphs of order $2p^2$

Classification of $2p^2$

Let p be a prime and let Γ be a connected pentavalent G-arc-transitive graph of order $2p^2$. Then G is isomorphic to one of the following graphs.

Γ	$\mathrm{Aut}(\Gamma)$	p
$\mathcal{CGD}_{p^2}^{[1]}$	$(\mathrm{Dih}(\mathbb{Z}_5^2) \rtimes F_{20})\mathbb{Z}_4$	$p = 5$
	$\mathrm{Dih}(\mathbb{Z}_p^2) \rtimes \mathbb{Z}_5$	$5 \mid (p-1)$
$\mathcal{CGD}_{p^2}^{[2]}$	$\mathrm{Dih}(\mathbb{Z}_p^2) \rtimes D_5$	$5 \mid (p \pm 1)$
$\mathcal{CD}_{p^2}$	$D_{p^2} \rtimes \mathbb{Z}_5$	$5 \mid (p-1)$

Proof for classification of $2p^2$

- For $p = 2$ or 3, $|V(\Gamma)| = 8$ or 18. There does not exists such a graph by McKay [25].
- Assume that $p \geq 5$. Then $A = \mathrm{Aut}(\Gamma)$ **has a semiregular subgroup L of order p^2 such that Γ is $N_A(L)$-arc-transitive.**
- For $L \cong \mathbb{Z}_p^2$, $\Gamma \cong \mathcal{CGD}_{p^2}^{[1]}$ or $\mathcal{CGD}_{p^2}^{[2]}$.
- For $L \cong \mathbb{Z}_{p^2}$, $\Gamma \cong \mathcal{CD}_{p^2}$ by Kwak et al. [14].

Classifications of pentavalent symmetric graphs of order $2p^3$

Classification of order $2p^3$

Let p be a prime and let Γ be a connected pentavalent G-arc-transitive graph of order $2p^3$. Then G is isomorphic to one of the following graphs.

Γ	$\mathrm{Aut}(\Gamma)$	p
FQ_4	$\mathbb{Z}_2^4 \rtimes \mathrm{S}_5$	$p=2$
$\mathcal{CD}_{p^3}$	$D_{p^3} \rtimes \mathbb{Z}_5$	$5 \mid (p-1)$
$\mathcal{CGD}_{p^3}$	$\mathrm{Dih}(\mathbb{Z}_p^3) \rtimes \mathrm{S}_5$	$p=5$
	$\mathrm{Dih}(\mathbb{Z}_p^3) \rtimes \mathbb{Z}_5$	$5 \mid (p-1)$
$\mathcal{CGD}^{[i]}_{p^2\times p}$ $(i=1,2,3)$	$\mathrm{Dih}(\mathbb{Z}_p^2 \times \mathbb{Z}_p) \rtimes \mathbb{Z}_5$	$5 \mid (p-1)$
$\mathcal{CN}^{[1]}_{2p^3}$	$(G_1(p) \rtimes \mathbb{Z}_2) \rtimes D_5$	$5 \mid (p \pm 1)$
$\mathcal{CN}^{[2]}_{2p^3}$	$(G_1(p) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_5$	$5 \mid (p-1)$

Proof for classification of order $2p^3$

- $\mathrm{Aut}(\Gamma)$ has a semiregular subgroup of order p^3, say P.
- Γ is $N_A(P)$-arc-transitive.
- Constructed graphs by considering regular covers of the Dipole Dip_5 with covering transformation group of order p^3.

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Thank you!