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Evropski socialni sklad

Vertex-Transitive Digraphs of Order p^5 are Hamiltonian

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Outline

Some concepts

Let Γ be a finite connected graph or digraph. A *Hamilton path* of Γ is a path going through all vertex of Γ. If a Hamilton path is closed, namely, a cycle, then we call this Hamilton path a *Hamilton cycle*. Γ is called *Hamiltonian* if it has a Hamilton Cycle.

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If $\mathrm{Aut}(\Gamma)$ acts transitively on the vertices set of Γ, then we call Γ a *vertex transitive graph (digraph)*. If $\mathrm{Aut}(\Gamma)$ contains a subgroup G acting regular on the vertices set of Γ, then we call Γ a *Cayley graph (digraph)* on G.

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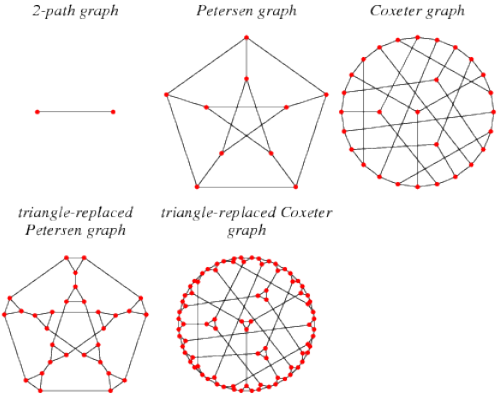
There are only four known nontrivial connected vertex-transitive graph that do not possess Hamilton cycles and none of them is a Cayley graph. Therefore there is a folklore conjecture as follows.

Conjecture A

Every connected Cayley graph with order greater than 2 is Hamiltonian.

The Known Nonhamiltonian V-T Graphs



V-T Graphs Which Are Known to Have a Hamilton Cycle

Connected vertex-transitive graphs of order kp, where $k \leq 4$, and p^i, where $i \leq 4$, and $2p^2$ (except for the Petersen graph and the Coxeter graph) contain a Hamilton cycle.

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The following contribution list.

B. Alspach (1979), Hamilton cycles in vertex-transitive graphs of order $2p$.
K. Kutnar, D. Marušič (2008), Hamiltonicity of vertex-transitive graphs of order $4p$.
D. Marušič (1985), Vertex-transitive graphs and digraphs of order p^k.
Y.Q. Chen (1998), On Hamiltonicity of vertex-transitive graphs and digraphs of order p^4.
D. Marušič (1987), Hamilton cycles in vertex symmetric graphs of order $2p^2$.

V-T Graphs Which Are Known to Have a Hamilton Path

A Hamilton path is known to exist in connected vertex-transitive graphs of order $5p$, $6p$ and order $10p$ removing some special cases.

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See the contributions below.

D. Marušič, T.D. Parsons (1982), Hamiltonian paths in vertex-symmetric graphs of order $5p$.
K. Kutnar, P. Šarl (2009), Hamilton paths and cycles in vertex-transitive graphs of order $6p$.
K. Kutnar, D. Marušič, C. Zhang (2012) Hamilton paths in vertex-transitive graphs of order $10p$.

Hamiltonicity of Cayley Graphs

Recall the following conjecture.

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Conjecture A is easy to prove for abelian groups and known to be true for hamiltonian groups (B. Alspach and Y. S. Qin, 2001), for metacyclic groups with respect to standard connection set (B. Alspach, 1989), for groups with a cyclic commutator subgroup of prime-power order (D. Marusic, E. Durnberger, K.Keating, and D.Witte, 1985), and for generalized dihedral groups of order a multiple of 4 (B. Alspach, C. C. Chen and M. Dean, 2010).

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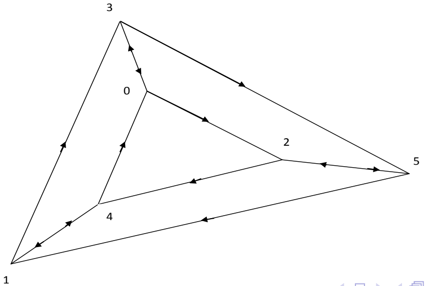

Hamiltonicity of V-T Digraphs of Prime Power Order

D.Witte (1986) proved a monumental theorem as follows.

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It is challenging to generalize Witte theorem to vertex-transitive digraphs of prime power order.

In this direction it was proved by Y. Q. Chen more than sixteen years ago that every connected vertex-transitive digraph of order p^4, where p is a prime, is Hamiltonian.

Main Theorems

In a manuscript still being refereed, I have proved the following results.

Theorem A (J-Y. Zhang,2014).

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Theorem B (J-Y. Zhang,2014).

Let Γ be a connected digraph of which the automorphism group contains a finite vertex-transitive subgroup G of prime power order. Let G' be the derived subgroup of G. Then Γ is Hamiltonian if one of the following two conditions hold:

1. *G' is generated by two elements;*
2. *G' is elementary abelian.*

Coset Digraphs

Let G be a finite group and H be a subgroup G. Let $G/H := \{gH \mid g \in G\}$ and $\Omega \subseteq \{HgH \mid g \in G - H\}$.

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The *coset digraph* $\Gamma = \mathrm{Cos}(G, H, \Omega)$ on G/H is defined as follows: the vertex set of Γ is G/H; the arc set of Γ is $\{(g_1H, g_2H) \mid Hg_1^{-1}g_2H \in \Omega\}$.

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The following proposition give a representation of a vertex-transitive digraph of prime power order.

Proposition A (Y. Q. Chen,1998).

If Γ is a vertex-transitive digraph of order p^n where p is a prime and n is a positive integer, then Γ admits a representation $\mathrm{Cos}(G, H, \Omega)$ such that G is a p-group, $H \bigcap Z(G) = \{1\}$ and $H < \Phi(G)$.

Coset Digraphs and Quotient Digraphs

Let $\Gamma = \mathrm{Cos}(G, H, \Omega)$ be a coset digraph on G/H. If K is a subgroup of G which contains H, then K induces quotient digraph Γ_K of Γ.

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Proposition B.

Let G be a finite group, H a subgroup of G, and $\Gamma = \mathrm{Cos}(G, H, \Omega)$ a coset digraph on G/H. Let K be a subgroup of G which contains H, and Γ_K the quotient digraph of Γ induced by K. Then $\Gamma_K \cong \mathrm{Cos}(G, K, \Lambda)$ where $\Lambda = \{KxK \mid HxH \in \Omega\}$.

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Method used in the proof: choosing K and lifting a Hamilton cycle of $\Gamma_K = \mathrm{Cos}(G, K, \Lambda)$ to a Hamilton cycle of $\Gamma = \mathrm{Cos}(G, H, \Omega)$.

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Lemma A

There exists an element $w \in Z(H)$ of order p such that $[G, w] \cap H = \{1\}$.

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Choose $K := [G, w]H$. Let $\Gamma = \mathrm{Cos}(G, H, \Omega)$ be a coset digraph on G/H and set $\Sigma = \mathrm{Cos}(G, K, \Lambda)$ where $\Lambda = \{KxK \mid HxH \in \Omega\}$.

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Lemma B (The lifting lemma)

Suppose that $\Sigma = \mathrm{Cos}(G, K, \Lambda)$ is Hamiltonian. Then $\Gamma = \mathrm{Cos}(G, H, \Omega)$ is Hamiltonian if one of the following two conditions holds:

1. $[G, w]$ *is generated by one or two elements;*
2. $[G, w]$ *is an elementary abelian group;*

Proof of Theorem A: If $|G : H| = p^5$, then $|[G, w]| \leq p^3$. Therefore $[G, w]$ can be generated by at most two elements or $[G, w]$ is elementary abelian. Thus the Lifting Lemma works.

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Proof of Theorem B: If G' is generated by two elements, then $[G, w]$ is generated by two elements and hence the Lifting Lemma works. If G' is elementary abelian, then we have the same conclusion.

I believe that every connected vertex transitive digraph of order p^n has a Hamilton cycle. By using Lemma B in this report and the classifications of groups of order p^4 and p^5, it will be not difficult to prove the same conclusion in Theorem A for p^6 and p^7.

Thank you very much !