Transitive combinatorial structures constructed from finite groups

Andrea Švob (asvob@math.uniri.hr)
Dean Crnković (deanc@math.uniri.hr)
Vedrana Mikulić Crnković (vmikulic@math.uniri.hr)

Department of Mathematics, University of Rijeka, Croatia

2014 PhD Summer School in Discrete Mathematics and SYGN IV, Rogla, Slovenia

July 2, 2014

REPUBLIKA SLOVENIJA
MINISTRSTVO ZA IZOBRAŽEVANJE, ZNANOST IN ŠPORT

Naložba v vašo prihodnost

An incidence structure is an ordered triple $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ where $\mathcal{P}$ and $\mathcal{B}$ are non-empty disjoint sets and $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{B}$.

The elements of the set $\mathcal{P}$ are called points, the elements of the set $\mathcal{B}$ are called blocks and $\mathcal{I}$ is called an incidence relation.

- An isomorphism from one incidence structure to other is a bijective mapping of points to points and blocks to blocks which preserves incidence.
- An isomorphism from an incidence structure $\mathcal{D}$ onto itself is called an automorphism of $\mathcal{D}$.
- The set of all automorphisms forms a group called the full automorphism group of $\mathcal{D}$ and is denoted by $Aut(\mathcal{D})$.

A $t-(v,k,\lambda)$ design is a finite incidence structure $\mathcal{D}=(\mathcal{P},\mathcal{B},\mathcal{I})$ satisfying the following requirements:

1. $|\mathcal{P}|=v$,
2. every element of $\mathcal{B}$ is incident with exactly k elements of $\mathcal{P}$,
3. every t elements of $\mathcal{P}$ are incident with exactly λ elements of $\mathcal{B}$.

A $2-(v,k,\lambda)$ design is called a block design.

Note that this definition allows $\mathcal{B}$ to be a multiset. If $\mathcal{B}$ is a set then $\mathcal{D}$ is called a simple design. If the design $\mathcal{D}$ consists of k copies of some simple design $\mathcal{D}'$ than $\mathcal{D}$ is nonsimple design and it is denoted $\mathcal{D}=k\mathcal{D}'$

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be a $t - (v, k, \lambda)$ design, with $0 \leq s \leq t$. $\mathcal{D}$ is also an $s - (v, k, \lambda_s)$ design where

$$\lambda_s \binom{k-s}{t-s} = \lambda \binom{v-s}{t-s}.$$

Every t-design is also an s-design for $s \leq t$.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{I})$ be a finite incidence structure. $\mathcal{G}$ is a graph if each element of $\mathcal{E}$ is incident with exactly two elements of $\mathcal{V}$. The elements of $\mathcal{V}$ are called vertices and the elements of $\mathcal{E}$ are called edges.

Two vertices u and v are called adjacent or neighbours if they are incident with the same edge. The number of neighbours of a vertex v is called the degree of v. If all the vertices of the graph $\mathcal{G}$ have the same degree k, then $\mathcal{G}$ is called k-regular.

A graph $\mathcal{G}$ is called a strongly regular graph with parameters (n, k, λ, μ), and denoted by $SRG(n, k, \lambda, \mu)$, if $\mathcal{G}$ is $k-$regular graph with n vertices and if any two adjacent vertices have λ common neighbours and any two non-adjacent vertices have μ common neighbours.

J. D. Key, J. Moori, Codes, Designs and Graphs from the Janko Groups J_1 and J_2, *J. Combin. Math. Combin. Comput.* 40 (2002), 143–159.

- The construction method of primitive symmetric designs and regular graphs for which a stabilizer of a point and a stabilizer of a block are conjugate.

D. Crnković, V. Mikulić, Unitals, projective planes and other combinatorial structures constructed from the unitary groups $U(3, q)$, $q = 3, 4, 5, 7$, *Ars Combin.* 110 (2013), 3–13.

- The construction method of primitive designs and regular graphs for which a stabilizer of a point and a stabilizer of a block are not necessarily conjugate.

- D. Crnković, V. Mikulić, A. Švob, On some transitive combinatorial structures constructed from the unitary group U(3,3), *J. Statist. Plann. Inference* 144 (2014), 19-40.

Theorem

Let G be a finite permutation group acting transitively on the sets Ω_1 and Ω_2 of size m and n, respectively. Let $\alpha \in \Omega_1$ and $\Delta_2 = \bigcup_{i=1}^{s} G_\alpha.\delta_i$, where $\delta_1, ..., \delta_s \in \Omega_2$ are representatives of distinct G_α-orbits. If $\Delta_2 \neq \Omega_2$ and

$$\mathcal{B} = \{g.\Delta_2 : g \in G\},$$

then $\mathcal{D}(G, \alpha, \delta_1, ..., \delta_s) = (\Omega_2, \mathcal{B})$ is a $1 - (n, |\Delta_2|, \frac{|G_\alpha|}{|G_{\Delta_2}|} \sum_{i=1}^{s} |G_{\delta_i}.\alpha|)$ design with $\frac{m \cdot |G_\alpha|}{|G_{\Delta_2}|}$ blocks. The group $H \cong G / \bigcap_{x \in \Omega_2} G_x$ acts as an automorphism group on $(\Omega_2, \mathcal{B})$, transitively on points and blocks of the design.

Corollary

If a group G acts transitively on the points and the blocks of a 1-design $\mathcal{D}$, then $\mathcal{D}$ can be obtained as described in the Theorem, i.e., such that Δ_2 is a union of G_α-orbits.

We can use the Theorem to construct $1-$design as follows. Let M be a finite group and H_1, H_2, and G be subgroups of M. G acts transitively on the class $ccl_G(H_i)$, $i = 1, 2$, by conjugation and

$$|ccl_G(H_1)| = [G : N_G(H_1)] = m,$$

$$|ccl_G(H_2)| = [G : N_G(H_2)] = n.$$

Let us denote the elements of $ccl_G(H_1)$ by $H_1^{g_1}, H_1^{g_2}, \dots, H_1^{g_m}$, and the elements of $ccl_G(H_2)$ by $H_2^{h_1}, H_2^{h_2}, \dots, H_2^{h_n}$.

We can construct a $1-$design such that:

- the point set of the design is $ccl_G(H_2)$,
- the block set is $ccl_G(H_1)$,
- the block $H_1^{g_i}$ is incident with the point $H_2^{h_j}$ if and only if $H_2^{h_j} \cap H_1^{g_i} \cong G_i,\ i = 1, \ldots, k$, where $\{G_1, ..., G_k\} \subset \{H_2^x \cap H_1^y \mid x, y \in G\}$.

We denote a $1-$design constructed in this way by $\mathcal{D}(G, H_2, H_1; G_1, ..., G_k)$.

Let M be a finite group and H and G be subgroups of M. One can construct regular graph in the following way:

- the vertex set of the graph is $ccl_G(H)$,
- the vertex H^{g_i} is adjacent to the vertex H^{g_j} if and only if $H^{g_i} \cap H^{g_j} \cong G_i$, $i = 1, \ldots, k$, where $\{G_1, ..., G_k\} \subset \{H^x \cap H^y \mid x, y \in G\}$.

We denote a regular graph constructed in this way by $\mathcal{G}(G, H; G_1, ..., G_k)$.

We consider transitive structures constructed from a simple group isomorphic to the unitary group $G \cong U(3,3)$. We describe structures constructed on the conjugacy classes of the maximal and second maximal subgroups of the group G.

Table: Maximal subgroups of the group $U(3,3)$ (up to conjugation)

Subgroup	Structure of the group	Size	Size of $G-$conjugacy class
M_1	$(E_9 : Z_3) : Z_8$	216	28
M_2	$L(2,7)$	168	36
M_3	$(Z_4 \times Z_4) : S_3$	96	63
M_4	$Z_4.S_4$	96	63

Table: Second maximal subgroups of the group $U(3,3)$ (up to conjugation)

Subgroup	Structure of the group	Size	Size of $G-$conjugacy class
H_1	$Ex_{27}^{+} : Z_4$	108	28
H_2	$E_4.A_4$	48	63
H_3	$Z_4.A_4$	48	63
H_4	$(Z_4 : Z_2) : Z_2$	32	189
H_5	S_4	24	252
H_6	$Z_3 : Z_8$	24	252
H_7	$Z_7 : Z_3$	21	288

In the following table we give a list of the 2-designs constructed on $G-$conjugacy classes of maximal and second maximal subgroups and some of their properties. The group G acts on all constructed designs, primitively on points and transitively but imprimitively on blocks.

Table: Transitive block designs constructed from the group $U(3,3)$

Block design $\mathcal{D}$	Parameters of $\mathcal{D}$	Simple design	Corresponding simple design	$Aut\mathcal{D}$
$\mathcal{D}(G, M_1, H_4; Z_8)$	(28, 4, 3)	no	(28, 4, 1)	$U(3,3):Z_2$
$\mathcal{D}(G, M_1, H_4; Z_4)$	(28, 8, 14)	yes		$U(3,3):Z_2$
$\mathcal{D}(G, M_1, H_4; Z_4, Z_8)$	(28, 12, 33)	no	(28, 12, 11)	$S(6,2)$
$\mathcal{D}(G, M_1, H_5; S_3)$	(28, 4, 4)	yes		$U(3,3):Z_2$
$\mathcal{D}(G, M_1, H_6; Z_8)$	(28, 3, 2)	yes		$U(3,3):Z_2$
$\mathcal{D}(G, M_1, H_6; Z_8, Z_3:Z_8)$	(28, 4, 4)	no	(28, 4, 1)	$U(3,3):Z_2$
$\mathcal{D}(G, M_1, H_7; Z_3)$	(28, 7, 16)	yes		$S(6,2)$
$\mathcal{D}(G, M_2, H_4; Z_2)$	(36, 16, 36)	no	(36, 16, 12)	$S(6,2)$
$\mathcal{D}(G, M_2, H_5; Z_2, S_3)$	(36, 16, 48)	no	(36, 16, 12)	$S(6,2)$
$\mathcal{D}(G, M_2, H_7; Z_3, Z_7:Z_3)$	(36, 15, 48)	no	(36, 15, 6)	$U(3,3):Z_2$
$\mathcal{D}(G, M_3, H_4; Z_2, Z_4, Z_2 \times Z_4, Z_4 \times Z_2, (Z_4:Z_2):Z_2)$	(63, 31, 45)	no	(63, 31, 15)	$U(3,3):Z_2$

- We did not obtain any strongly regular graph from $G-$conjugacy classes of second maximal subgroups (whose $G-$normalizer is not a maximal subgroup).

- The group $U(3,3)$ has 190 maximal subgroups, and has four distinct $U(3,3)-$conjugacy classes of the maximal subgroups M_1, M_2, M_3, M_4.
- We consider structures constructed on the conjugacy classes of the maximal subgroups of the group $U(3,3)$ under the action of the four not conjugate maximal subgroups.
- We do not need to consider conjugacy classes of all maximal subgroups, we can eliminate some of them.

- Finally, after elimination, we got 7 maximal subgroups of the $U(3,3)$ which are not conjugate under the action of the group M_1,
- 11 maximal subgroups of the $U(3,3)$ which are not conjugate under the action of the group M_2,
- 11 maximal subgroups of the $U(3,3)$ which are not conjugate under the action of the group M_3,
- 14 maximal subgroups of the $U(3,3)$ which are not conjugate under the action of the group M_4.

Table: Block designs constructed from the group $U(3,3)$, from the conjugacy classes of maximal subgroups under the action of the maximal subgroups

Block design $\mathcal{D}$	Parameters of $\mathcal{D}$	Simple design	Corresponding simple design	$Aut\mathcal{D}$
$\mathcal{D}(M_2, (Z_4 \times Z_4) : S_3, (Z_4 \times Z_4) : S_3; D_8 : Z_2)$	$(7, 3, 1)$	yes		$L(2, 7)$
$\mathcal{D}(M_2, (Z_4 \times Z_4) : S_3, (Z_4 \times Z_4) : S_3; E_4)$	$(7, 3, 3)$	no	$(7, 3, 1)$	$L(2, 7)$
$\mathcal{D}(M_2, (Z_4 \times Z_4) : S_3, (Z_9 : Z_3) : Z_8; Z_8)$	$(7, 3, 4)$	no	$(7, 3, 1)$	$L(2, 7)$
$\mathcal{D}(M_2, L(2, 7), (Z_4 \times Z_4) : S_3; S_3)$	$(7, 3, 4)$	yes		$L(2, 7)$
$\mathcal{D}(M_1, Z_4.S_4, L(2, 7); Z_4)$	$(9, 3, 3)$	no	$(9, 3, 1)$	$(E_9 : Z_2).S_4$
$\mathcal{D}(M_1, Z_4.S_4, Z_4.S_4; Z_4 \times Z_4)$	$(9, 4, 9)$	no	$(9, 4, 3)$	$(E_9 : D_8).Z_2$
$\mathcal{D}(M_4, (Z_4 \times Z_4) : S_3, (Z_4 \times Z_4) : S_3; S_3)$	$(16, 6, 2)$	yes		$E_{16} : S_6$
$\mathcal{D}(M_1, L(2, 7), L(2, 7); D_8)$	$(36, 15, 6)$	yes		$U(3, 3) : Z_2$
$\mathcal{D}(M_1, (Z_4 \times Z_4) : S_3, (Z_4 \times Z_4) : S_3; E_4, S_3)$	$(36, 15, 6)$	yes		$U(4, 2) : Z_2$

Table: Strongly regular graphs constructed from the group $U(3,3)$ from the conjugacy classes of maximal subgroups under the action of the maximal subgroups

Graph $\mathcal{G}$	Parameters of $\mathcal{G}$	$Aut\mathcal{G}$
$\mathcal{G}(M_4, (Z_4 \times Z_4) : S_3; S_3)$	$(16, 6, 2, 2)$	$(Z_4 \times Z_4) : D_{12}$
$\mathcal{G}(M_1, (Z_4 \times Z_4) : S_3; E_4, D_8 : Z_2)$	$(27, 10, 1, 5)$	$U(4,2) : Z_2$
$\mathcal{G}(M_2, (Z_4 \times Z_4) : S_3; S_3)$	$(28, 12, 6, 4)$	S_8
$\mathcal{G}(M_1, L(2,7); S_4)$	$(36, 14, 4, 6)$	$U(3,3) : Z_2$
$\mathcal{G}(M_1, (Z_4 \times Z_4) : S_3; E_4, D_8 : Z_2)$	$(36, 15, 6, 6)$	$U(4,2) : Z_2$

Up to $U(3,3)$-conjugation, the group $U(3,3)$ has 7 second maximal subgroups.
We consider structures constructed on the conjugacy classes of the second maximal subgroups of the group $U(3,3)$ under the action of the maximal subgroups M_1, M_2, M_3 and M_4.

- After elimination, we got 17 second maximal subgroups of the $U(3,3)$ which are not conjugate under the action of the group M_1,
- 26 maximal subgroups of the $U(3,3)$ which are not conjugate under the action of the group M_2,
- 29 maximal subgroups of the $U(3,3)$ which are not conjugate under the action of the group M_3,
- 36 maximal subgroups of the $U(3,3)$ which are not conjugate under the action of the group M_4.

Table: Block designs constructed from the group $U(3,3)$ from the conjugacy classes of maximal and second maximal subgroups under the action of the maximal subgroups

Block design $\mathcal{D}$	Parameters of $\mathcal{D}$	Simple design	Corresponding simple design	$Aut\mathcal{D}$
$\mathcal{D}(M_2, M_2^3, H_2^{14}; Z_2 \times Z_4)$	$(7,3,1)$	yes		$L(2,7)$
$\mathcal{D}(M_2, M_2^3, H_2^1; Z_2 \times Z_4, Z_2)$	$(7,3,3)$	no	$(7,3,1)$	$L(2,7)$
$\mathcal{D}(M_2, M_2^3, H_2^6; Z_2)$	$(7,3,4)$	no	$(7,3,1)$	$L(2,7)$
$\mathcal{D}(M_2, M_2^3, H_2^4; E_4, D_8)$	$(7,3,6)$	no	$(7,3,1)$	$L(2,7)$
$\mathcal{D}(M_2, M_2^7, H_2^{25}; I)$	$(7,3,8)$	no	$(7,3,1)$	$L(2,7)$
$\mathcal{D}(M_2, M_2^3, H_2^{22}; I)$	$(7,3,8)$	no	$(7,3,4)$	$L(2,7)$
$\mathcal{D}(M_2, M_2^3, H_2^{19}; Z_8, Z_2)$	$(7,3,12)$	no	$(7,3,1)$	$L(2,7)$
$\mathcal{D}(M_2, M_2^7, H_2^8; E_4)$	$(7,3,12)$	no	$(7,3,4)$	$L(2,7)$
$\mathcal{D}(M_2, M_2^7, H_2^{23}; Z_3)$	$(7,3,24)$	no	$(7,3,1)$	$L(2,7)$
$\mathcal{D}(M_1, M_1^3, H_1^5; Z_4)$	$(9,3,3)$	no	$(9,3,1)$	$E_9 : (SL(2,3) : Z_2)$
$\mathcal{D}(M_1, H_1^8, H_1^7; Z_4, I)$	$(9,3,9)$	no	$(9,3,1)$	$E_9 : (SL(2,3) : Z_2)$
$\mathcal{D}(M_1, M_1^3, H_1^7; Z_4, I)$	$(9,4,18)$	no	$(9,4,3)$	$(E_9 : D_8).Z_2$
$\mathcal{D}(M_4, M_4^{14}, H_4^{31}; Z_3)$	$(16,5,8)$	yes		$E_{16} : S_6$
$\mathcal{D}(M_4, M_4^3, H_4^{10}; Z_3)$	$(16,6,2)$	yes		$E_{16} : S_6$
$\mathcal{D}(M_4, M_4^{10}, H_4^{32}; Z_3)$	$(16,6,4)$	no	$(16,6,2)$	$E_{16} : S_6$
$\mathcal{D}(M_4, M_4^{10}, H_4^{12}; S_3, Z_2)$	$(16,6,6)$	no	$(16,6,2)$	$E_{16} : S_6$
$\mathcal{D}(M_4, H_4^{10}, H_4^7; Z_3, Z_2)$	$(16,6,6)$	yes		$E_{16}.S_4$
$\mathcal{D}(M_4, H_4^{13}, H_4^5; I)$	$(16,6,6)$	yes		$E_{16}.S_4$
$\mathcal{D}(M_4, M_4^{10}, H_4^{31}; Z_3)$	$(16,6,12)$	no	$(16,6,2)$	$E_{16} : S_6$
$\mathcal{D}(M_1, M_1^2, H_1^5; Z_2, Z_2 \times Z_4)$	$(36,15,6)$	yes		$U(4,2) : Z_2$
$\mathcal{D}(M_1, M_1^5, H_1^{15}; Z_7 : Z_3, Z_3)$	$(36,15,12)$	no	$(36,15,6)$	$U(3,3) : Z_2$
$\mathcal{D}(M_1, M_1^5, H_1^{14}; Z_7 : Z_3, Z_3)$	$(36,15,36)$	no	$(36,15,6)$	$U(3,3) : Z_2$

Table: Strongly regular graphs constructed from the group $U(3,3)$ from the conjugacy classes of second maximal subgroups under the action of the maximal subgroups

Graph $\mathcal{G}$	Parameters of $\mathcal{G}$	$Aut\mathcal{G}$
$\mathcal{G}(M_4, H_4^{10}; Z_3)$	$(16, 6, 2, 2)$	$(Z_4 \times Z_4) : D_{12}$
$\mathcal{G}(M_1, H_1^{1}; I, Z_4)$	$(27, 10, 1, 5)$	$U(4,2) : Z_2$
$\mathcal{G}(M_1, H_1^{6}; I, E_4)$	$(36, 15, 6, 6)$	$U(4,2) : Z_2$

We consider transitive structures constructed from a simple group G isomorphic to the symplectic group $S(6, 2)$. We describe structures constructed on the conjugacy classes of the maximal and second maximal subgroups of the group G.

Table: Maximal subgroups of the group $S(6,2)$ (up to conjugation)

Subgroup	Structure of the subgroup	Size	Size of G-conjugacy class
M_8	$U(4,2):Z_2$	51840	28
M_7	S_8	40320	36
M_6	$E_{32}:S_6$	23040	63
M_5	$U(3,3):Z_2$	12096	120
M_4	$E_{64}:L(3,2)$	10752	135
M_3	$((E_{16}:Z_2)\times E_4):(S_4\times S_4)$	4608	315
M_2	$S_3\times S_6$	4320	336
M_1	$L(2,8):Z_3$	1512	960

Table: Second maximal subgroups of the group $S(6,2)$ (up to conjugation)

Subgroup	Structure of the group	Size	Size of G-conjugacy class
H_1	$(E_{16} : A_5) : Z_2$	1920	378
H_2	$((Z_2 \times D_8) : Z_2) : (S_3 \times S_3)$	1152	1260
H_3	$Z_2 \times S_6$	1440	1008
H_4	$(E_9 : Z_3) : GL(2,3)$	1296	1120
H_5	$E_{27} : (Z_2 \times S_4)$	1296	1120
H_6	$(S_4 \times S_4) : Z_2$	1152	1260
H_7	S_7	5040	288
H_8	$E_8 : (Z_2 \times S_4)$	384	3780
H_9	$S_5 \times S_3$	720	2016
H_{10}	$PSL(32) : Z_2$	336	4320
H_{11}	$(E_{32} : A_5) : Z_2$	3840	378
H_{12}	$Z_2 \times ((E_{16} : A_5) : Z_2)$	3840	378
H_{13}	$Z_2 \times ((S_4 \times S_4) : Z_2)$	2304	630
H_{14}	$E_{32} : (Z_2 : S_4)$	1536	945
H_{15}	$E_8 : (D_8 \times S_4)$	1536	945
H_{16}	$Z_2 \times S_6$	1440	1008
H_{17}	$(E_9 : Z_3) : QD_{16}$	432	3360
H_{18}	$(SL(23) : Z_4) : Z_2$	192	7560
H_{19}	$E_4 : (Z_2 \times S_4)$	192	7560
H_{20}	$E_{32} : (Z_2 \times S_4)$	1536	945

Table: Second maximal subgroups of the group $S(6,2)$ (up to conjugation) (continued from the previous page)

Subgroup	Structure of the group	Size	Size of G-conjugacy class
H_{21}	$(E_{64} : Z_7) : Z_3$	1344	1080
H_{22}	$E_8.PSL(32)$	1344	1080
H_{23}	$E_8 : PSL(32)$	1344	1080
H_{24}	$Z_2 \times S_3 \times S_4$	288	5040
H_{25}	$S_5 \times S_3$	720	2016
H_{26}	$((S_3 \times S_3) : Z_2) \times S_3$	432	3360
H_{27}	$Z_2 \times S_4 \times S_3$	288	5040
H_{28}	$(E_8 : Z_7) : Z_3$	168	8640
H_{29}	$(Z_9 : Z_3) : Z_2$	54	26880
H_{30}	$E_{21} : Z_2$	42	34560

- We describe 2−designs and strongly regular graphs obtained from G−conjugacy classes of the maximal and second maximal subgroups.
- The group G acts transitively on all constructed designs.

Table: Transitive block designs constructed from the group $S(6, 2)$, $v = 28$

Block design $\mathcal{D}$	Parameters of $\mathcal{D}$	Simple design	Corresponding simple design	$\mathrm{Aut}(\mathcal{D})$
$\mathcal{D}(G, M_8, H_6; P^1_{8,6})$	$(28, 12, 110)$	no	$(28, 12, 11)$	$S(6, 2)$
$\mathcal{D}(G, M_8, H_7; P^1_{8,7})$	$(28, 7, 16)$	yes		$S(6, 2)$
$\mathcal{D}(G, M_8, H_8; P^1_{8,8})$	$(28, 4, 60)$	no	$(28, 4, 5)$	$S(6, 2)$
$\mathcal{D}(G, M_8, H_9; P^1_{8,9})$	$(28, 3, 16)$	yes		$S(6, 2)$
$\mathcal{D}(G, M_8, H_9; P^2_{8,9})$	$(28, 10, 240)$	no	$(28, 10, 40)$	$S(6, 2)$
$\mathcal{D}(G, M_8, H_9; P^1_{8,9}, P^2_{8,9})$	$(28, 13, 416)$	yes		$S(6, 2)$
$\mathcal{D}(G, M_8, H_{11}; P^1_{8,11})$	$(28, 12, 66)$	no	$(28, 12, 11)$	$S(6, 2)$
$\mathcal{D}(G, M_8, H_{14}; P^1_{8,14})$	$(28, 12, 165)$	no	$(28, 12, 11)$	$S(6, 2)$
$\mathcal{D}(G, M_8, H_{15}; P^1_{8,15})$	$(28, 4, 15)$	no	$(28, 4, 5)$	$S(6, 2)$
$\mathcal{D}(G, M_8, H_{15}; P^2_{8,15})$	$(28, 8, 70)$	yes		$S(6, 2)$
$\mathcal{D}(G, M_8, H_{16}; P^1_{8,16})$	$(28, 10, 120)$	no	$(28, 10, 40)$	$S(6, 2)$
$\mathcal{D}(G, M_8, H_{16}; P^2_{8,16})$	$(28, 12, 176)$	no	$(28, 12, 11)$	$S(6, 2)$
$\mathcal{D}(G, M_8, H_{16}; P^3_{8,16})$	$(28, 6, 40)$	yes		$S(6, 2)$
$\mathcal{D}(G, M_8, H_{18}; P^1_{8,18})$	$(28, 4, 120)$	no	$(28, 4, 5)$	$S(6, 2)$

Table: Transitive block designs constructed from the group $S(6,2)$, $v = 28$ (continued from the previous page)

Block design $\mathcal{D}$	Parameters of $\mathcal{D}$	Simple design	Corresponding simple design	$\mathrm{Aut}(\mathcal{D})$
$\mathcal{D}(G, M_8, H_{19}; P^1_{8,19})$	$(28, 12, 660)$	no	$(28, 12, 11)$	$S(6,2)$
$\mathcal{D}(G, M_8, H_{24}; P^1_{8,24})$	$(28, 6, 200)$	yes		$S(6,2)$
$\mathcal{D}(G, M_8, H_{24}; P^2_{8,24})$	$(28, 4, 80)$	no	$(28, 4, 5)$	$S(6,2)$
$\mathcal{D}(G, M_8, H_{24}; P^1_{8,24}, P^2_{8,24})$	$(28, 10, 600)$	no	$(28, 10, 40)$	$S(6,2)$
$\mathcal{D}(G, M_8, H_{26}; P^1_{8,26})$	$(28, 9, 320)$	yes		$S(6,2)$
$\mathcal{D}(G, M_8, H_{26}; P^1_{8,26}, P^2_{8,26})$	$(28, 10, 400)$	no	$(28, 10, 40)$	$S(6,2)$
$\mathcal{D}(G, M_8, H_{27}; P^1_{8,27})$	$(28, 4, 80)$	yes		$S(6,2)$
$\mathcal{D}(G, M_8, H_{27}; P^2_{8,27}, P^3_{8,27})$	$(28, 12, 880)$	no	$(28, 12, 11)$	$S(6,2)$
$\mathcal{D}(G, M_8, H_{27}; P^4_{8,27})$	$(28, 12, 880)$	yes		$S(6,2)$
$\mathcal{D}(G, M_8, H_1; P^1_{8,1})$	$(28, 10, 45)$	yes		$S(6,2)$
$\mathcal{D}(G, M_8, H_2; P^1_{8,2})$	$(28, 3, 10)$	yes		$S(6,2)$
$\mathcal{D}(G, M_8, H_2; P^1_{8,2}, P^2_{8,2})$	$(28, 4, 20)$	no	$(28, 4, 5)$	$S(6,2)$
$\mathcal{D}(G, M_8, H_3; P^1_{8,3}, P^2_{8,3})$	$(28, 13, 208)$	yes		$S(6,2)$
$\mathcal{D}(G, M_8, H_{30}; P^1_{8,30})$	$(28, 7, 1920)$	no	$(28, 7, 16)$	$S(6,2)$

Table: Transitive block designs constructed from the group $S(6,2)$, $v = 36$

Block design $\mathcal{D}$	Parameters of $\mathcal{D}$	Simple design	Corresponding simple design	$\mathrm{Aut}(\mathcal{D})$
$\mathcal{D}(G, M_7, H_1; P^1_{7,1})$	(36, 16, 72)	no	(36, 16, 12)	$S(6,2)$
$\mathcal{D}(G, M_7, H_2; P^1_{7,2})$	(36, 12, 132)	no	(36, 12, 33)	$S(6,2)$
$\mathcal{D}(G, M_7, H_5; P^1_{7,5})$	(36, 9, 64)	yes		$S(6,2)$
$\mathcal{D}(G, M_7, H_8; P^1_{7,8})$	(36, 3, 18)	yes		$S(6,2)$
$\mathcal{D}(G, M_7, H_8; P^1_{7,8}, P^2_{7,8})$	(36, 4, 36)	no	(36, 4, 9)	$S(6,2)$
$\mathcal{D}(G, M_7, H_8; P^3_{7,8})$	(36, 8, 168)	no	(36, 8, 6)	$S(6,2)$
$\mathcal{D}(G, M_7, H_8; P^2_{7,8}, P^3_{7,8})$	(36, 9, 216)	yes		$S(6,2)$
$\mathcal{D}(G, M_7, H_8; P^1_{7,8}, P^3_{7,8}$	(36, 11, 330)	yes		$S(6,2)$
$\mathcal{D}(G, M_7, H_8; P^1_{7,8}, P^2_{7,8}, P^3_{7,8})$	(36, 12, 396)	no	(36, 12, 33)	$S(6,2)$
$\mathcal{D}(G, M_7, H_9; P^1_{7,9})$	(36, 5, 32)	yes		$S(6,2)$
$\mathcal{D}(G, M_7, H_9; P^1_{7,9}, P^2_{7,9})$	(36, 6, 48)	no	(36, 6, 8)	$S(6,2)$
$\mathcal{D}(G, M_7, H_{10}; P^1_{7,10})$	(36, 14, 624)	yes		$S(6,2)$
$\mathcal{D}(G, M_7, H_{10}; P^1_{7,10}, P^2_{7,10})$	(36, 15, 720)	yes		$S(6,2)$
$\mathcal{D}(G, M_7, H_{14}; P^1_{7,14})$	(36, 8, 42)	no	(36, 8, 6)	$S(6,2)$
$\mathcal{D}(G, M_7, H_{14}; P^2_{7,14})$	(36, 12, 99)	yes		$S(6,2)$
$\mathcal{D}(G, M_7, H_{14}; P^3_{7,14})$	(36, 16, 180)	no	(36, 16, 12)	$S(6,2)$
$\mathcal{D}(G, M_7, H_{15}; P^1_{7,15})$	(36, 12, 99)	no	(36, 12, 33)	$S(6,2)$
$\mathcal{D}(G, M_7, H_{15}; P^2_{7,15})$	(36, 8, 42)	yes		$S(6,2)$
$\mathcal{D}(G, M_7, H_{16}; P^1_{7,16})$	(36, 6, 24)	no	(36, 6, 8)	$S(6,2)$

Table: Transitive block designs constructed from the group $S(6,2)$, $v = 36$ (continued from the previous page)

Block design $\mathcal{D}$	Parameters of $\mathcal{D}$	Simple design	Corresponding simple design	$\mathrm{Aut}(\mathcal{D})$
$\mathcal{D}(G, M_7, H_{16}; P^2_{7,16})$	$(36, 10, 72)$	yes		$S(6,2)$
$\mathcal{D}(G, M_7, H_{16}; P^1_{7,16}, P^2_{7,16})$	$(36, 16, 192)$	no	$(36, 16, 12)$	$S(6,2)$
$\mathcal{D}(G, M_7, H_{18}; P^1_{7,18})$	$(36, 12, 792)$	no	$(36, 12, 33)$	$S(6,2)$
$\mathcal{D}(G, M_7, H_{19}; P^1_{7,19})$	$(36, 16, 720)$	no	$(36, 16, 12)$	$S(6,2)$
$\mathcal{D}(G, M_7, H_{19}; P^2_{7,19})$	$(36, 12, 396)$	no	$(36, 12, 99)$	$S(6,2)$
$\mathcal{D}(G, M_7, H_{20}; P^1_{7,20})$	$(36, 4, 9)$	yes		$S(6,2)$
$\mathcal{D}(G, M_7, H_{21}; P^1_{7,21})$	$(36, 8, 48)$	no	$(36, 8, 6)$	$S(6,2)$
$\mathcal{D}(G, M_7, H_{24}; P^1_{7,24})$	$(36, 6, 120)$	no	$(36, 6, 8)$	$S(6,2)$
$\mathcal{D}(G, M_7, H_{24}; P^2_{7,24})$	$(36, 12, 528)$	no	$(36, 12, 33)$	$S(6,2)$
$\mathcal{D}(G, M_7, H_{24}; P^1_{7,24}, P^2_{7,24})$	$(36, 18, 1224)$	yes		$S(6,2)$
$\mathcal{D}(G, M_7, H_{24}; P^3_{7,24})$	$(36, 18, 1224)$	yes		$S(6,2)$
$\mathcal{D}(G, M_7, H_{26}; P^1_{7,26})$	$(36, 6, 80)$	no	$(36, 6, 8)$	$S(6,2)$
$\mathcal{D}(G, M_7, H_{26}; P^2_{7,26})$	$(36, 3, 16)$	yes		$S(6,2)$
$\mathcal{D}(G, M_7, H_{26}; P^1_{7,26}, P^2_{7,26})$	$(36, 9, 192)$	no	$(36, 9, 64)$	$S(6,2)$

Table: Transitive block designs constructed from the group $S(6,2)$, $v = 36$ (continued from the previous page)

Block design $\mathcal{D}$	Parameters of $\mathcal{D}$	Simple design	Corresponding simple design	$\mathrm{Aut}(\mathcal{D})$
$\mathcal{D}(G, M_7, H_{27}; P^1_{7,27})$	$(36, 4, 48)$	yes		$S(6,2)$
$\mathcal{D}(G, M_7, H_{27}; P^2_{7,27})$	$(36, 12, 528)$	yes		$S(6,2)$
$\mathcal{D}(G, M_7, H_{27}; P^2_{7,27}, P^3_{7,27})$	$(36, 14, 728)$	yes		$S(6,2)$
$\mathcal{D}(G, M_7, H_{27}; P^2_{7,27}, P^1_{7,27})$	$(36, 16, 960)$	no	$(36, 16, 12)$	$S(6,2)$
$\mathcal{D}(G, M_7, H_{27}; P^2_{7,27}, P^1_{7,27}, P^3_{7,27})$	$(36, 18, 1224)$	no	$(36, 18, 153)$	$S(6,2)$
$\mathcal{D}(G, M_7, H_{27}; P^4_{7,27})$	$(36, 18, 1224)$	no	$(36, 18, 153)$	$S(6,2)$
$\mathcal{D}(G, M_7, H_{28}; P^1_{7,28})$	$(36, 8, 384)$	no	$(36, 8, 6)$	$S(6,2)$
$\mathcal{D}(G, M_7, H_3; P^1_{7,3})$	$(36, 15, 168)$	yes		$S(6,2)$
$\mathcal{D}(G, M_7, H_6; P^1_{7,6})$	$(36, 16, 120)$	no	$(36, 16, 12)$	$S(6,2)$
$\mathcal{D}(G, M_7, H_6; P^1_{7,6}, P^2_{7,6})$	$(36, 18, 153)$	yes		$S(6,2)$
$\mathcal{D}(G, M_7, H_6; P^3_{7,6})$	$(36, 18, 153)$	yes		$S(6,2)$
$\mathcal{D}(G, M_7, H_{23}; P^1_{7,23})$	$(36, 7, 36)$	yes		$S(6,2)$
$\mathcal{D}(G, M_7, H_{29}; P^1_{7,29})$	$(36, 9, 1536)$	no	$(36, 9, 64)$	$S(6,2)$
$\mathcal{D}(G, M_7, H_{30}; P^1_{7,30})$	$(36, 14, 4992)$	no	$36, 14, 624$	$S(6,2)$
$\mathcal{D}(G, M_7, H_{30}; P^1_{7,30}, P^2_{7,30})$	$(36, 15, 5760)$	no	$(36, 15, 720)$	$S(6,2)$

Table: Transitive block designs constructed from the group $S(6,2)$, $v = 63, 120, 378$

Block design $\mathcal{D}$	Parameters of $\mathcal{D}$	Simple design	Corresponding simple design	$\mathrm{Aut}(\mathcal{D})$
$\mathcal{D}(G, M_6, H_1; P^1_{6,1}, P^2_{6,1}, P^3_{6,1})$	$(63, 31, 90)$	no	$(63, 31, 15)$	$PGL(6,2)$
$\mathcal{D}(G, M_6, H_3; P^1_{6,3}, P^2_{6,3})$	$(63, 31, 240)$	no	$(63, 31, 15)$	$PGL(6,2)$
$\mathcal{D}(G, M_6, H_6; P^1_{6,6}, P^2_{6,6}, P^3_{6,6})$	$(63, 31, 150)$	no	$(63, 31, 15)$	$PGL(6,2)$
$\mathcal{D}(G, M_6, H_{14}; P^1_{6,14}, P^2_{6,14}, P^3_{6,14})$	$(63, 31, 225)$	no	$(63, 31, 15)$	$PGL(6,2)$
$\mathcal{D}(G, M_6, H_{19}; P^1_{6,19}, P^2_{6,19}, P^3_{6,19})$	$(63, 31, 900)$	no	$(63, 31, 15)$	$PGL(6,2)$
$\mathcal{D}(G, M_6, H_{27}; P^1_{6,27}, P^2_{6,27}, P^3_{6,27}, P^4_{6,27})$	$(63, 31, 1200)$	no	$(63, 31, 15)$	$PGL(6,2)$
$\mathcal{D}(G, M_5, H_{10}; P^1_{5,10}, P^2_{5,10})$	$(120, 35, 360)$	yes		$O^+(8,2) : Z_2$
$\mathcal{D}(G, H_1, H_{12}; P^1_{1,12}, P^2_{1,12}, P^3_{1,12})$	$(378, 117, 36)$	yes		$O(7,3) : Z_2$

Table: Strongly regular graphs constructed from the group $S(6,2)$ from the conjugacy classes of the second maximal subgroups

Graph $\mathcal{G}$	Parameters of $\mathcal{G}$	$\mathrm{Aut}(\mathcal{G})$
$\mathcal{G}(G_2, H_{12}; P_{12}^1, P_{12}^2)$	$(378, 52, 26, 4)$	S_{28}
$\mathcal{G}(G_2, H_{12}; P_{12}^1, P_{12}^3, P_{12}^4)$	$(378, 117, 36, 36)$	$O_7(3) : Z_2$
$\mathcal{G}(G_2, H_{13}; P_{13}^1, P_{13}^2)$	$(630, 68, 34, 4)$	S_{36}
$\mathcal{G}(G_2, H_4; P_4^1, P_4^2, P_4^3, P_4^4)$	$(1120, 390, 146, 130)$	$O_8^+(3).D_8$

Thank you for your attention!

Transitive combinatorial structures constructed from finite groups

Andrea Švob (asvob@math.uniri.hr)
Department of Mathematics, University of Rijeka, Croatia

REPUBLIKA SLOVENIJA
MINISTRSTVO ZA IZOBRAŽEVANJE, ZNANOST IN ŠPORT
Naložba v vašo prihodnost