The Classification of Minimal non-core-2 2-groups with Almost Maximal Class

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Definition

Let G be a finite group and $H \leq G$. H is called a core-n subgroup of G if $|H : H_G| \leq n$ where $H_G = \bigcap_{g \in G} H^g$ is the core of H in G.

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Definition

G is called a core-n group if each subgroup of G is a core-n subgroup.

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Let G be a finite p-group with order p^n, where p is a prime. G is called an almost maximal class if $c(G) = n - 2$.

Remark: In this paper, G always is not abelian. And the terminology is general.

Introduction

For a group G and its subgroup H,

$$1 \leq H_G \leq H \trianglelefteq N_G(H) \leq G.$$

When any subgroup H of G such that $N_G(H) = H$, G is called Dedekind group. Certainly, at the same time, $H_G = H$.
So the core-n group can be seen as some generalized Dedekind group.

Background

Buckley, Lennox, Neumann, Smith and Wiegold studied n-core p-group in 1995[1]. Their paper concerned the maximal abelian normal subgroup index's bounder for the core-p p-group.
After that, J.C. Lennox, H.Smith, J.Wiegold, Y.Berkovich, Z.Janko, M.Y. Xu and some others gave some contributions about core-p p-group in [2, 3, 4, 5, 6].

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By the classification maximal class p-group, one can check they are the core-p p-group.
In this paper, we decide the minimal non-core-2 2-groups with almost maximal class.

Theorem

Let G be a non-core-2 2-group of order 2^n with almost maximal class. Then G must be one of following groups.

(I) $\langle a, b, c, d | c^{2^{n-3}} = d^2 = b^2 = [d, c] = [b, d] = 1, c^b = c^{-1}, b^a = bc, d^a = dc^{2^{n-4}}, c^a = c^{-1+2^{n-4}}, a^2 = dc^{2^{n-5}} \rangle, (n \geq 6)$;

(II) $\langle a, b, c, d | c^{2^{n-3}} = d^2 = [d, c] = [b, d] = 1, c^b = c^{-1+2^{n-4}}, b^2 = d, b^a = bc, d^a = dc^{2^{n-4}}, c^a = c, a^2 = c^{2^{n-5}} \rangle, (n \geq 6)$;

(III) $\langle a, b, c, d | c^{2^{n-3}} = d^2 = a^2 = [d, c] = [b, d] = 1, c^b = c^{-1+2^{n-4}}, b^2 = d, b^a = bc, d^a = dc^{2^{n-4}}, c^a = c^{-1} \rangle, (n \geq 5)$;

(IV) $\langle a, b, c, d | c^2 = d^2 = b^2 = a^4 = [d, c] = [b, d] = [b, c] = [d, a] = 1, [b, a] = c, [c, a] = d \rangle (n = 5)$;

(V) $\langle a, b, c, d | c^2 = d^2 = b^2 = [d, c] = [b, d] = [b, c] = [d, a] = 1, a^4 = d, [b, a] = c, [c, a] = d \rangle (n = 5)$;

(VI) $\langle c, b, a | c^{2^{n-2}} = b^2 = a^2 = 1, c^b = c^{-1}, c^a = c^{-1+2^{n-3}}, [b, a] = 1 \rangle, (n \geq 5)$.

Remark

1. I, II and III: $d(G) = 2$ *and* G' *is cyclic where* $n \geq 5$*;*
2. IV and V: $d(G) = 2$ *and* G' *is not cyclic where n=5;*
3. VI: $d(G) = 3$ *where* $n \geq 5$.

Lemma

Let G be a groups of order 2^n where $n \leq 4$ or $c(G) = n - 1$. Then G is core-2 2-group.

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This lemma means both of small order and the maximal class 2-group are the core-2 2-group. So we only consider $n \geq 5$ in the following lemma.

Lemma

Let G be a minimal non-core-p p-groups of order p^n where $n \geq 4$, then $d(G) \leq 4$.

Outline of the Proof.
Let $K < G$ such that $|K : K_G| \geq p^2$. Since G is minimal, $K_G = 1$.
We can choose a subgroup $H \leq K$ with order p^2 such that $H_G = 1$, $H \bigcap Z(G) = 1$ and $H\Phi(G) < G$.
Case 1. If $|G : H\Phi(G)| = p$, then $|G : \Phi(G)| \leq p^3$;
Case 2. Otherwise $|G : H\Phi(G)| \geq p^2$, then can get $|G : \Phi(G)| \leq p^4$. □

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Lemma

Let G be a minimal non-core-2 2-groups of order 2^n with almost maximal class where $n > 4$, then $d(G) \leq 3$.

Proof.

Since G is almost maximal class, $|G/G'| = 2^3$ and $G' \leq \Phi(G)$. Then $|G:\Phi(G)| \leq 2^3$. So $d(G) = 2$ or 3. □

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Remark

We classify the group according to $d(G) = 2$ or 3.

Lemma

Let G be a minimal non-core-2 2-groups of order 2^n with almost maximal class where $n \geq 7$, $d(G) = 2$. Then G' is cyclic.

Outline of the proof.
Firstly, we claim that
when $n = 6$, if $d(G') \geq 2$(it means that G' is not cyclic) and $d(G) \geq 2$, then G is not a core-2 2-group.

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when $n = 6$, if $d(G') \geq 2$(it means that G' is not cyclic) and $d(G) \geq 2$, then G is not a core-2 2-group.
Secondly, assume G' is not cyclic. Since G is a minimal non-core-2 2-group, then $\bar{G} = G/G_5$ is core-2 2-group with order 2^6 such that $d(\bar{G}') \geq 2$, $|\bar{G}/\Phi(\bar{G})| = 4$.
This is a contradiction with above claiming result.

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So G' is cyclic. □

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Lemma

Let G be a minimal non-core-2 2-groups of order 2^n where $n \geq 5$, $d(G) = 2$ and G' is cyclic. Then G must be one of following groups.

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(II) $\langle a, b, c, d | c^{2^{n-3}} = d^2 = [d, c] = [b, d] = 1, c^b = c^{-1+2^{n-4}}, b^2 = d, b^a = bc, d^a = dc^{2^{n-4}}, c^a = c, a^2 = c^{2^{n-5}} \rangle, (n \geq 6)$;

(III) $\langle a, b, c, d | c^{2^{n-3}} = d^2 = a^2 = [d, c] = [b, d] = 1, c^b = c^{-1+2^{n-4}}, b^2 = d, b^a = bc, d^a = dc^{2^{n-4}}, c^a = c^{-1} \rangle, (n \geq 5)$.

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Since G is a minimal non-core-2 2-group, we can take subgroup H of order 4 and $H_G = 1$.

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Set $G' = \langle c_0 \rangle$ and $O(c_0) = 2^{n-3}$. Thus $H \bigcap G' = 1$, $M = G' \rtimes H \lessdot G$.

For M is a core-2 2-group, take $L = \{1, d\} \lhd M$ and $L \leq H$.

Then $\langle c_0 \rangle \times L \lessdot M$.

Thus $H = \langle L, b \rangle, G = \langle M, a \rangle$ where $b^2 \in L$.

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$\forall x \in G$, since $[d, x] \equiv 1 \pmod{G'}$, then $d^x = d \cdot c_0^j, 0 \leq j < 2^{n-3}$.
For $O(d) = O(d^x) = O(dc_0^j) = 2$ and $n \geq 5$, then

$$2^{n-4} | j \text{ and } [d, x] \equiv 1 \pmod{G_3 = \langle c_0^2 \rangle}.$$

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Thus $d \in \Phi(G), \Phi(G) = \langle c_0, d \rangle$, and $G = \langle a, b \rangle$.

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Thus $d \in \Phi(G), \Phi(G) = \langle c_0, d \rangle$, and $G = \langle a, b \rangle$.
Then

$$G' = \langle c_0 \rangle = \langle [b,a], G_3 \rangle, [b,a] = c_0^{\delta_0}, (\delta_0, 2) = 1.$$

Since $H_G = 1$, $\{1, d\} \neq \{1, d\}^a \subset \Omega_1(\Phi(G)) = \{1, d, dc_0^{2^{n-4}}, c_0^{2^{n-4}}\}$.
But $d^a \equiv d \pmod{G' = \langle c_0 \rangle}$, $d^a = dc_0^{2^{n-4}}$ and $a^2 \in \Phi(G)$.

Since $H_G = 1$, $\{1, d\} \neq \{1, d\}^a \subset \Omega_1(\Phi(G)) = \{1, d, dc_0^{2^{n-4}}, c_0^{2^{n-4}}\}$.
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Thus

$$c_0^a = c_0^{\delta_3}, a^2 = d^{\delta_4} c_0^{\delta_5}, \delta_3 \in \{\pm 1 + 2^{n-4}, \pm 1\}, \delta_4 \in \{0, 1\}, 0 \leq \delta_5 < 2^{n-4}.$$

Case 1. $n \geq 6$.
If $H \lhd M$, then$M = \langle c_0 \rangle \times H$, and

$$O(b) = O(b^a) = O(bc_0^{\delta_0}) = o(c_0^{\delta_0}) = o(c_0) \geq 8 \neq o(b) \leq 4.$$

A contradiction.
Thus $[c_0, b] = c_0^{\delta_1 - 1} \neq 1$.
Since $b^2 \in L$, $\delta_1 \in \{\pm 1 + 2^{n-4}, -1\}$. $b^2 = d^{\delta_2}, \delta_2 \in \{0, 1\}$.

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$$d^a = dc_0^{2^{n-4}}, c_0^a = c_0^{\delta_3}, a^2 = d^{\delta_4} c_0^{\delta_5} \rangle.$$

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$$d^a = dc_0^{2^{n-4}}, c_0^a = c_0^{\delta_3}, a^2 = d^{\delta_4} c_0^{\delta_5} \rangle.$$

Taking $c = c_0^{\delta_0}$, then

$$G = \langle a, b, c, d | c^{2^{n-3}} = d^2 = [d, c] = [b, d] = 1, c^b = c^{\delta_1}, b^2 = d^{\delta_2},$$
$$b^a = bc, d^a = dc^{2^{n-4}}, c^a = c^{\delta_3}, a^2 = d^{\delta_4} c^{\delta_5} \rangle.$$

Since $G/G' \cong \mathbb{Z}_4 \times \mathbb{Z}_2$,

$$\delta_2^2 + \delta_4^2 \neq 0. \tag{1}$$

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Since

$$\begin{aligned} (a^2)^a &= (d^{\delta_4} c^{\delta_5})^a = (dc^{2^{n-4}})^{\delta_4} c^{\delta_3 \cdot \delta_5} = d^{\delta_4} c^{2^{n-4} \cdot \delta_4 + \delta_3 \cdot \delta_5} \\ &= a^2 = d^{\delta_4} c^{\delta_5} \end{aligned}$$

,

$$2^{n-4}\delta_4 + (\delta_3 - 1)\delta_5 \equiv 0 \pmod{2^{n-3}}. \tag{2}$$

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Since

$$\begin{aligned} b^{a^2} &= b^{(d^{\delta_4} c^{\delta_5})} = b^{c^{\delta_5}} = b[b, c^{\delta_5}] = bc^{(1-\delta_1)\delta_5} \\ &= (b^a)^a = (bc)^a = bc^{\delta_3+1}, \end{aligned}$$

$$\delta_3 + 1 \equiv (1 - \delta_1)\delta_5 \pmod{2^{n-3}}. \tag{3}$$

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$$\delta_3 + 1 \equiv (1-\delta_1)\delta_5 \pmod{2^{n-3}}. \qquad (3)$$

Since

$$\begin{aligned} (b^2)^a &= (b^a)^2 = (bc)^2 = b^2 c^b c = b^2 c^{\delta_1+1} = d^{\delta_2} c^{\delta_1+1} \\ &= (d^{\delta_2})^a = (d^a)^{\delta_2} = d^{\delta_2} c^{2^{n-4}\delta_2}, \end{aligned}$$

$$2^{n-4}\delta_2 \equiv \delta_1 + 1 \pmod{2^{n-3}} \qquad (4)$$

.

By solving equations,
$\delta_3 = -1 + 2^{n-4}$, $\delta_5 = 2^{n-5} + 2^{n-4}\delta_5'$. Replacing $ad^{\delta_5'}$ with a, we can get $\delta_5 = 2^{n-5}$ and G is type(I).

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$\delta_3 = 1$,$\delta_5 = 1 + 2^{n-5} + 2^{n-4}\delta_5'$. Replacing $ad^{\delta_5'}$ with a, we can get $\delta_5 = 2^{n-5}$ and G is type (II).

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$\delta_3 = -1 + 2^{n-4}$ or -1, $\delta_5 = 2^{n-4}\delta_5'$, $\delta_3 = -1$. 用 $ad^{\delta_5'}$. Replacing a with δ_5', we can get $\delta_5 = 0$ and G is type (III).
Other cases do not occur or same as above type.

Case 2. n=5.

$$G = \langle a, b, c, d | c^4 = d^2 = [d, c] = [b, d] = 1, c^b = c^{\delta_1}, b^2 = d^{\delta_2}, b^a = bc, d^a = d$$

$$c^a = c^{\delta_3}, a^2 = d^{\delta_4} c^{\delta_5} \rangle, \quad \delta_1, \delta_3 \in \{\pm 1\}, \delta_2, \delta_4 \in \{0, 1\}, \delta_5 \in \{0, \pm 1, 2\}.$$

As above discussion, we can get a type (III) group.

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As above discussion, we can get a type (III) group.
We can check the three type group are not isomorphic and minimal non-core-2 2-group with almost maximal class.

□

Lemma

Let G be a minimal non-core-2 2-groups of order 2^n where $n \geq 5$, $d(G) = 2$ and G' is not cyclic. Then G must be one of following groups.
(IV)$\langle a, b, c, d | c^2 = d^2 = b^2 = a^4 = [d, c] = [b, d] = [b, c] = [d, a] = 1, [b, a] = c, [c, a] = d\rangle$ *($n = 5$); or*
(V)$\langle a, b, c, d | c^2 = d^2 = b^2 = [d, c] = [b, d] = [b, c] = [d, a] = 1, a^4 = d, [b, a] = c, [c, a] = d\rangle$ *($n = 5$);*

Proof. By above lemma, since G' is not cyclic, then $4 < n \leq 6$.
Case 1. n=5.
Since G' is not cyclic, $G' \cong \mathbb{Z}_2 \times \mathbb{Z}_2$,$G/G' \cong \mathbb{Z}_4 \times \mathbb{Z}_2$.
Without loss generality, we can assume $G = \langle a, b\rangle$, and $a^4, b^2 \in G', [a, b] = c \in G' - G_3, G_3 = \langle d\rangle$.

Then

$$G = \langle a, b, c, d | c^2 = d^2 = [d, c] = [b, d] = [d, a] = 1, [b, a] = c,$$
$$[c, a] = d^{\delta_1}, [c, b] = d^{\delta_2}, a^4 = c^{\delta_3} d^{\delta_4}, b^2 = c^{\delta_5} d^{\delta_6} \rangle$$

where $\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6 \in \{0, 1\}$.

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where $\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6 \in \{0, 1\}$.
Since G is almost maximal class,

$$\begin{aligned}
[G_2, G] = \langle [c, a], [c, b], G_3 \rangle &\Rightarrow \delta_1 + \delta_2 \geq 1 \\
(b^2)^a = (b^a)^2 &\Rightarrow d^{\delta_1 \delta_5} = d^{\delta_2} \\
(a^4)^a = a^4 &\Rightarrow d^{\delta_1 \delta_3} = 1 \\
(b^2)^b = b^2 &\Rightarrow d^{\delta_2 \delta_5} = 1 \\
(a^4)^b = (a^b)^4 &\Rightarrow d^{\delta_2 \delta_3} = 1.
\end{aligned}$$

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$$G = \langle a, b, c, d | c^2 = d^2 = [d, c] = [b, d] = [d, a] = 1, [b, a] = c,$$

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where $\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6 \in \{0, 1\}$.
Since G is almost maximal class,

$$\begin{array}{rcl}
[G_2, G] = \langle [c, a], [c, b], G_3 \rangle & \Rightarrow & \delta_1 + \delta_2 \geq 1 \\
(b^2)^a = (b^a)^2 & \Rightarrow & d^{\delta_1 \delta_5} = d^{\delta_2} \\
(a^4)^a = a^4 & \Rightarrow & d^{\delta_1 \delta_3} = 1 \\
(b^2)^b = b^2 & \Rightarrow & d^{\delta_2 \delta_5} = 1 \\
(a^4)^b = (a^b)^4 & \Rightarrow & d^{\delta_2 \delta_3} = 1.
\end{array}$$

Then $\delta_1 = 1, \delta_2 = \delta_3 = \delta_5 = 0, \delta_4, \delta_6 \in \{0, 1\}$.

(i) If $\delta_4 = 0, \delta_6 = 0$,then G is type (V);
(ii)If $\delta_4 = 0, \delta_6 = 1$, replacing ba^2 with b, then G is type(V);
(iii)If $\delta_4 = 1, \delta_6 = 0$, then G is type (IV);
(iv)If $\delta_4 = 1$ and $\delta_6 = 1$, this case G is core-2 2-group.

Case 2. n=6. It is similarity to $n = 5$.

$\square$

Lemma

Let G be a minimal non-core-2 2-groups of order 2^n where $n \geq 5$, $d(G) = 3$. Then G must be
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Without loss generality, we can assume $H = \langle b, a\rangle$

$$G = \langle a, b, c | c^{2^{n-2}} = b^2 = a^2 = 1, c^b = c^{-1}, c^a = c^{-1+2^{n-3}}, [b, a] = 1\rangle.$$

We can check G is an inner core-2 2-group and G/N is a core-2 2-group where $N \leq Z(G)$ and $|N| = 2$.
Then G is a minimal non-core-2 2-group. □

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By above lemma, we can get the main result. □

Furthermore, we can also get all the inner core-2 2-groups.

Thank you very much!

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