Finite-dimensional irreducible modules for an even subalgebra of $U_q(\mathfrak{sl}_2)$

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Introduction

Fix a field $\mathbb{F}$ and fix $0 \neq q \in \mathbb{F}$ not a root of unity.

In this talk, we consider a subalgebra of the $\mathbb{F}$-algebra $U_q(\mathfrak{sl}_2)$.

The Lie algebra $\mathfrak{sl}_2$

The Lie algebra $\mathfrak{sl}_2$ consists of the 2 x 2 matrices over $\mathbb{F}$ with trace 0.

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$$[x, y] = xy - yx.$$

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$\mathfrak{sl}_2$ has a basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Observe that

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

The algebras $U(\mathfrak{sl}_2)$ and $U_q(\mathfrak{sl}_2)$

The universal enveloping algebra $U(\mathfrak{sl}_2)$ is the associative algebra defined by generators e, f, h and relations

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The quantum enveloping algebra $U_q(\mathfrak{sl}_2)$ is the associative algebra defined by generators e, f, k, k^{-1} and relations

$$kk^{-1} = k^{-1}k = 1,$$

$$kek^{-1} = q^2 e, \qquad kfk^{-1} = q^{-2} f,$$

$$ef - fe = \frac{k - k^{-1}}{q - q^{-1}}.$$

Equitable presentation for $U_q(\mathfrak{sl}_2)$

In 2006, Ito, Terwilliger, and Weng showed that $U_q(\mathfrak{sl}_2)$ has a presentation in generators $x, y^{\pm 1}, z$ and relations

$$yy^{-1} = y^{-1}y = 1,$$
$$\frac{qxy - q^{-1}yx}{q - q^{-1}} = 1,$$
$$\frac{qyz - q^{-1}zy}{q - q^{-1}} = 1$$
$$\frac{qzx - q^{-1}xz}{q - q^{-1}} = 1.$$

This presentation is called the **equitable presentation** for $U_q(\mathfrak{sl}_2)$.

Connections with $U_q(\mathfrak{sl}_2)$

$U_q(\mathfrak{sl}_2)$ and its equitable presentation have connections with:

- Q-polynomial distance-regular graphs (Worawannotai, 2012),
- Leonard pairs (Alnajjar, 2011),
- Tridiagonal pairs (Ito/Terwilliger, 2007),
- the q-Tetrahedron algebra (Ito/Terwilliger 2007, Funk-Neubauer 2009, Miki 2010),
- the universal Askey-Wilson algebra (Terwilliger, 2011).

A basis for $U_q(\mathfrak{sl}_2)$

Lemma (Terwilliger, 2011)

The following is a basis for the $\mathbb{F}$-vector space $U_q(\mathfrak{sl}_2)$:

$$x^r y^s z^t \qquad r, t \in \mathbb{N}, \quad s \in \mathbb{Z}.$$

The algebra $\mathcal{A}$

Define $\mathcal{A}$ to be the $\mathbb{F}$-subspace of $U_q(\mathfrak{sl}_2)$ spanned by

$$x^r y^s z^t \qquad r, s, t \in \mathbb{N}, \quad r + s + t \text{ even.}$$

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Lemma (Bockting-Conrad and Terwilliger, 2013)

$\mathcal{A}$ is a subalgebra of $U_q(\mathfrak{sl}_2)$.

The elements ν_x, ν_y, ν_z

The relations from the equitable presentation for $U_q(\mathfrak{sl}_2)$ can be reformulated as:

$$q(1 - xy) = q^{-1}(1 - yx),$$
$$q(1 - yz) = q^{-1}(1 - zy),$$
$$q(1 - zx) = q^{-1}(1 - xz).$$

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$$q(1 - zx) = q^{-1}(1 - xz).$$

We denote these elements ν_x, ν_y, ν_z respectively.

Observe that $\nu_x, \nu_y, \nu_z \in \mathcal{A}$.

Generators for $\mathcal{A}$

Proposition (Bockting-Conrad and Terwilliger, 2013)

The $\mathbb{F}$-algebra $\mathcal{A}$ is generated by ν_x, ν_y, ν_z.

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In the same paper, Bockting-Conrad and Terwilliger posed the problem of finding a presentation for $\mathcal{A}$ in generators ν_x, ν_y, ν_z.

Relations involving ν_x, ν_y, ν_z

Proposition

In $U_q(\mathfrak{sl}_2)$, the elements ν_x, ν_y, ν_z satisfy

$$q^3\nu_x^2\nu_y - (q+q^{-1})\nu_x\nu_y\nu_x + q^{-3}\nu_y\nu_x^2 = (q^2-q^{-2})(q-q^{-1})\nu_x,$$
$$q^3\nu_y^2\nu_z - (q+q^{-1})\nu_y\nu_z\nu_y + q^{-3}\nu_z\nu_y^2 = (q^2-q^{-2})(q-q^{-1})\nu_y,$$
$$q^3\nu_z^2\nu_x - (q+q^{-1})\nu_z\nu_x\nu_z + q^{-3}\nu_x\nu_z^2 = (q^2-q^{-2})(q-q^{-1})\nu_z,$$

and

$$q^{-3}\nu_y^2\nu_x - (q+q^{-1})\nu_y\nu_x\nu_y + q^3\nu_x\nu_y^2 = (q^2-q^{-2})(q-q^{-1})\nu_y,$$
$$q^{-3}\nu_z^2\nu_y - (q+q^{-1})\nu_z\nu_y\nu_z + q^3\nu_y\nu_z^2 = (q^2-q^{-2})(q-q^{-1})\nu_z,$$
$$q^{-3}\nu_x^2\nu_z - (q+q^{-1})\nu_x\nu_z\nu_x + q^3\nu_z\nu_x^2 = (q^2-q^{-2})(q-q^{-1})\nu_x.$$

Relations involving ν_x, ν_y, ν_z

Proposition (AGL)

In $U_q(\mathfrak{sl}_2)$, the elements ν_x, ν_y, ν_z satisfy

$$\nu_x \frac{q\nu_y\nu_z - q^{-1}\nu_z\nu_y}{q-q^{-1}} = \nu_x - q^{-2}\nu_y - q^2\nu_z + \frac{q^2\nu_y\nu_z - q^{-2}\nu_z\nu_y}{q-q^{-1}},$$

$$\frac{q\nu_y\nu_z - q^{-1}\nu_z\nu_y}{q-q^{-1}}\nu_x = \nu_x - q^{2}\nu_y - q^{-2}\nu_z + \frac{q^2\nu_y\nu_z - q^{-2}\nu_z\nu_y}{q-q^{-1}},$$

and the relations obtained from these by cyclically permuting $\nu_x \to \nu_y \to \nu_z \to \nu_x$.

A presentation for $\mathcal{A}$

Theorem

The $\mathbb{F}$-algebra $\mathcal{A}$ is isomorphic to the $\mathbb{F}$-algebra defined by generators ν_x, ν_y, ν_z and the 12 relations from the previous two propositions.

We now turn our attention to the representation theory of $\mathcal{A}$.

First, we recall the representation theory of $U_q(\mathfrak{sl}_2)$.

Representation theory of $U_q(\mathfrak{sl}_2)$

For $n \in \mathbb{N}, \varepsilon \in \{1, -1\}$, there exists an irreducible $U_q(\mathfrak{sl}_2)$-module $L(n, \varepsilon)$ of dimension n which has a basis $\{v_i\}_{i=0}^{n}$ such that

$$\begin{aligned} \varepsilon x.v_i &= q^{2i-n} v_i + (q^n - q^{2i-2-n}) v_{i-1} \quad (1 \le i \le n), \\ \varepsilon x.v_0 &= q^{-n} v_0, \\ \varepsilon y.v_i &= q^{n-2i} v_i \quad (0 \le i \le n), \\ \varepsilon z.v_i &= q^{2i-n} v_i + (q^{-n} - q^{2i+2-n}) v_{i+1} \quad (0 \le i \le n-1), \\ \varepsilon z.v_n &= q^n v_n. \end{aligned}$$

Moreover, every finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$-module is isomorphic to some $L(n, \varepsilon)$.

Induced modules of $\mathcal{A}$

Observe that $L(n, \epsilon)$ has an induced $\mathcal{A}$-module structure.

For $n \in \mathbb{N}$, the $\mathcal{A}$-modules $L(n, 1)$ and $L(n, -1)$ are isomorphic.

We denote by $L(n)$ the common $\mathcal{A}$-module structure of $L(n, 1)$ and $L(n, -1)$.

Facts about $L(n)$

- $L(n)$ is irreducible as an $\mathcal{A}$-module.
- The actions of ν_x, ν_y, ν_z on $L(n)$ are nilpotent.
- The actions of x^2, y^2, z^2 on $L(n)$ are diagonalizable.

Facts about finite-dimensional irreducible $\mathcal{A}$-modules

What about arbitrary finite-dimensional irreducible $\mathcal{A}$-modules?

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Lemma (AGL)

Let V be a finite-dimensional irreducible $\mathcal{A}$-module. Then the actions of ν_x, ν_y, ν_z on V are nilpotent.

Facts about finite-dimensional irreducible $\mathcal{A}$-modules

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Representation theory of $\mathcal{A}$

Theorem (AGL)

Let V be a finite-dimensional irreducible $\mathcal{A}$-module. Then V is isomorphic to $L(n)$ for some $n \in \mathbb{N}$.

Future work

- Investigate the induced $\mathcal{A}$-modules from the $U_q(\mathfrak{sl}_2)$ modules related to tridiagonal pairs, the q-tetrahedron algebra, etc.
- Are there any naturally arising $\mathcal{A}$-modules other than those induced by an existing $U_q(\mathfrak{sl}_2)$-module?

Thank you!