Billiard arrays and finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$-modules

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Overview

In this talk we will describe the notion of a **Billiard Array**.

This is a triangular array of one-dimensional subspaces of a finite-dimensional vector space, subject to several conditions that specify which sums are direct.

We will use Billiard Arrays to characterize the finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$-modules, for q not a root of unity.

Motivation: $U_q(\mathfrak{sl}_2)$ and its modules

In order to motivate things, we recall the quantum algebra $U_q(\mathfrak{sl}_2)$.

We will use the **equitable presentation**.

Let $\mathbb{F}$ denote a field. Fix a nonzero $q \in \mathbb{F}$ that is not a root of unity.

The definition of $U_q(\mathfrak{sl}_2)$

Definition

Let $U_q(\mathfrak{sl}_2)$ denote the associative $\mathbb{F}$-algebra with generators $x, y^{\pm 1}, z$ and relations $yy^{-1} = y^{-1}y = 1$,

$$\frac{qxy - q^{-1}yx}{q - q^{-1}} = 1,$$

$$\frac{qyz - q^{-1}zy}{q - q^{-1}} = 1,$$

$$\frac{qzx - q^{-1}xz}{q - q^{-1}} = 1.$$

The $x, y^{\pm 1}, z$ are called the **equitable generators** for $U_q(\mathfrak{sl}_2)$.

The elements ν_x, ν_y, ν_z

The defining relations for $U_q(\mathfrak{sl}_2)$ can be reformulated as follows:

$$
\begin{aligned}
q(1 - yz) &= q^{-1}(1 - zy), \\
q(1 - zx) &= q^{-1}(1 - xz), \\
q(1 - xy) &= q^{-1}(1 - yx).
\end{aligned}
$$

Denote these common values by ν_x, ν_y, ν_z respectively.

How x, y, z are related to ν_x, ν_y, ν_z

The x, y, z are related to ν_x, ν_y, ν_z as follows:

$$
\begin{aligned}
x\nu_y &= q^2 \nu_y x, & \qquad x\nu_z &= q^{-2}\nu_z x,\\
y\nu_z &= q^2 \nu_z y, & \qquad y\nu_x &= q^{-2}\nu_x y,\\
z\nu_x &= q^2 \nu_x z, & \qquad z\nu_y &= q^{-2}\nu_y z.
\end{aligned}
$$

For the rest of this talk, fix an integer $N \geq 1$.

Let V denote an irreducible $U_q(\mathfrak{sl}_2)$-module, with dimension $N+1$.

The x, y, z act on V as follows.

Each of x, y, z is **diagonalizable** on V. Moreover there exists $\varepsilon \in \{1, -1\}$ such that for each of x, y, z the eigenvalues on V are $\{\varepsilon q^{N-2i}\}_{i=0}^{N}$. This ordering and its inversion will be called **standard**.

The scalar ε is called the **type** of V.

Replacing x, y, z by $\varepsilon x, \varepsilon y, \varepsilon z$ the type becomes 1.

From now on, assume that V has type 1.

$U_q(\mathfrak{sl}_2)$-modules, cont.

The ν_x, ν_y, ν_z act on V as follows.

Each of ν_x, ν_y, ν_z is **nilpotent** on V.

Moreover, for $\rho \in \{x, y, z\}$ the subspace $\nu_\rho^i V$ has dimension $N - i + 1$ for $0 \leq i \leq N$, and $\nu_\rho^{N+1} V = 0$.

Decompositions and flags

In order to clarify how x, y, z and ν_x, ν_y, ν_z act on V, we use the following concepts.

By a **decomposition** of V we mean a sequence $\{V_i\}_{i=0}^N$ of one-dimensional subspaces of V whose direct sum is V.

Example

For each of x, y, z the sequence of eigenspaces (in standard order) is a decomposition of V, said to be **standard**.

By a **flag** on V we mean a sequence $\{U_i\}_{i=0}^N$ of subspaces for V such that $U_{i-1} \subseteq U_i$ for $1 \leq i \leq N$ and U_i has dimension $i+1$ for $0 \leq i \leq N$.

Example

Each of

$$\{\nu_x^{N-i} V\}_{i=0}^N, \qquad \{\nu_y^{N-i} V\}_{i=0}^N, \qquad \{\nu_z^{N-i} V\}_{i=0}^N$$

is a flag on V, said to be **standard**.

From decompositions to flags

Given a decomposition $\{V_i\}_{i=0}^N$ of V we construct a flag on V as follows.

Define $U_i = V_0 + \cdots + V_i$ for $0 \leq i \leq N$. Then the sequence $\{U_i\}_{i=0}^N$ is a flag on V.

This flag is said to be **induced** by the decomposition $\{V_i\}_{i=0}^N$.

From flags to decompositions

Let $\{U_i\}_{i=0}^N$ and $\{U'_i\}_{i=0}^N$ denote flags on V.

These flags are called **opposite** whenever $U_i \cap U'_j = 0$ if $i + j < N$ $(0 \leq i, j \leq N)$.

The flags $\{U_i\}_{i=0}^N$ and $\{U'_i\}_{i=0}^N$ are opposite if and only if there exists a decomposition $\{V_i\}_{i=0}^N$ of V that induces $\{U_i\}_{i=0}^N$ and whose inversion $\{V_{N-i}\}_{i=0}^N$ induces $\{U'_i\}_{i=0}^N$.

In this case $V_i = U_i \cap U'_{N-i}$ for $0 \leq i \leq N$.

So we say that the decomposition $\{V_i\}_{i=0}^N$ is **induced** by the opposite flags $\{U_i\}_{i=0}^N$ and $\{U'_i\}_{i=0}^N$.

The standard flags and decompositions

Theorem

For our $U_q(\mathfrak{sl}_2)$-module V, the three standard flags are mutually opposite.

The standard flags are related to the standard decompositions in the following way.

Theorem

For our $U_q(\mathfrak{sl}_2)$-module V,

(i) *each standard decomposition of V induces a standard flag on V;*

(ii) *each ordered pair of distinct standard flags on V induces a standard decomposition of V.*

A problem in linear algebra

The above theorems suggest a problem in linear algebra.

Consider the three standard flags on our $U_q(\mathfrak{sl}_2)$-module V.

From these flags we can recover the standard decompositions of V, and from them the original $U_q(\mathfrak{sl}_2)$-module structure.

So these flags should be related in a special way, from a linear algebraic point of view.

The problem is to describe this relationship.

This is what we will do, for the rest of the talk.

The set Δ_N

Recall the natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$.

Definition

Let Δ_N denote the set consisting of the three-tuples of natural numbers whose sum is N. Thus

$$\Delta_N = \{(r, s, t) \mid r, s, t \in \mathbb{N}, \quad r + s + t = N\}.$$

The set Δ_N

We arrange the elements of Δ_N in a triangular array.

For $N = 4$, the array looks as follows after deleting all punctuation:

$$\begin{array}{ccccccccc} & & & & 040 & & & & \\ & & & 130 & & 031 & & & \\ & & 220 & & 121 & & 022 & & \\ & 310 & & 211 & & 112 & & 013 & \\ 400 & & 301 & & 202 & & 103 & & 004 \end{array}$$

An element in Δ_N is called a **location**.

The lines in Δ_N

				040				
			130		031			
		220		121		022		
	310		211		112		013	
400		301		202		103		004

In the above array, each horizontal row consists of the locations with the same middle coordinate.

Call the horizontal rows 2-**lines**.

The 1-**lines** and 3-**lines** are similarly defined.

By a **line** we mean a 1-line or 2-line or 3-line.

040

130 031

220 121 022

310 211 112 013

400 301 202 103 004

In the above array, each interior location is adjacent to six other locations.

By a 3-**clique** we mean a set of three mutually adjacent locations.

There are two kinds of 3-cliques: Δ (**white**) and ∇ (**black**).

We now define a Billiard Array.

Let V denote a vector space over $\mathbb{F}$ with dimension $N+1$.

Definition

By a **Billiard Array on** V we mean a function B that assigns to each location $\lambda \in \Delta_N$ a 1-dimensional subspace of V (denoted B_λ) such that:

(i) for each line L in Δ_N the sum $\sum_{\lambda \in L} B_\lambda$ is direct;

(ii) for each white 3-clique C in Δ_N the sum $\sum_{\lambda \in C} B_\lambda$ is not direct.

We say that B is **over** $\mathbb{F}$. We call N the **diameter** of B.

Comments on Billiard Arrays

Let B denote a Billiard Array on V.

It turns out that the function B is injective.

We view B as an arrangement of one-dimensional subspaces of V into a triangular array, with the subspace B_λ at location λ for all $\lambda \in \Delta_N$.

Thus the subspaces B_λ are the "billiards" in the array.

Here is our plan for the rest of the talk:

(i) Classify the Billiard Arrays up to isomorphism.

(ii) Describe what the Billiard Arrays have to do with 3-tuples of mutually opposite flags.

(iii) Use Billiard Arrays to explain what is special about the three standard flags for a finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$-module of type 1.

The classification of Billiard Arrays; preliminaries

Our next goal is to classify the Billiard Arrays up to isomorphism.

Lemma

Let λ, μ, ν denote locations in Δ_N that form a white 3-clique. Then the subspace $B_\lambda + B_\mu + B_\nu$ is equal to each of

$$B_\lambda + B_\mu, \qquad B_\mu + B_\nu, \qquad B_\nu + B_\lambda.$$

This subspace has dimension 2.

Corollary

Let λ, μ, ν denote locations in Δ_N that form a white 3-clique. Then each of B_λ, B_μ, B_ν is contained in the sum of the other two.

Among the lines in Δ_N, three are on the boundary.

Lemma

Let L denote a boundary line of Δ_N. Then

$$V = \sum_{\lambda \in L} B_\lambda \qquad \text{(direct sum)}.$$

Shortly we will classify the Billiard Arrays up to isomorphism.

To prepare for this, we explain what isomorphism means in this context.

Definition

Let V' denote a vector space over $\mathbb{F}$ with dimension $N+1$. Let B' denote a Billiard Array on V'. By an **isomorphism of Billiard Arrays from B to B'** we mean an $\mathbb{F}$-linear bijection $V \to V'$ that sends $B_\lambda \mapsto B'_\lambda$ for all $\lambda \in \Delta_N$. The Billiard Arrays B and B' are called **isomorphic** whenever there exists an isomorphism of Billiard Arrays from B to B'.

We now describe the notion of an affine brace.

Definition

Let λ, μ, ν denote locations in Δ_N that form a white 3-clique. By an **affine brace** (or **abrace**) for this clique, we mean a set of vectors

$$u \in B_\lambda, \qquad v \in B_\mu, \qquad w \in B_\nu$$

that are not all zero, and $u + v + w = 0$. (In fact each of u, v, w is nonzero).

Here is an example of an abrace.

Example

Let λ, μ, ν denote locations in Δ_N that form a white 3-clique. Pick any nonzero vectors

$$u \in B_\lambda, \qquad v \in B_\mu, \qquad w \in B_\nu.$$

The vectors u, v, w are linearly dependent. So there exist scalars a, b, c in $\mathbb{F}$, not all zero, such that $au + bv + cw = 0$. The vectors au, bv, cw form an abrace for the clique.

Affine braces have the following property.

Lemma

Let λ, μ, ν denote locations in Δ_N that form a white 3-clique. Then each nonzero vector in B_λ is contained in a unique abrace for this clique.

The braces for a Billiard Array

We have been discussing affine braces.

We now consider a variation on this concept, called a brace.

Definition

Let λ, μ denote adjacent locations in Δ_N. Note that there exists a unique location $\nu \in \Delta_N$ such that λ, μ, ν form a white 3-clique. We call ν the **completion** of the pair λ, μ.

Definition

Let λ, μ denote adjacent locations in Δ_N. By a **brace** for λ, μ we mean a set of nonzero vectors

$$u \in B_\lambda, \qquad v \in B_\mu$$

such that $u + v \in B_\nu$. Here ν denotes the completion of λ, μ.

The braces for a Billiard Array, cont.

Braces have the following property.

Lemma

Let λ, μ denote adjacent locations in Δ_N. Each nonzero vector in B_λ is contained in a unique brace for λ, μ.

The maps $\tilde{B}_{\lambda,\mu}$

We now define some maps $\tilde{B}_{\lambda,\mu}$.

Definition

Let λ, μ denote adjacent locations in Δ_N. We define an $\mathbb{F}$-linear map $\tilde{B}_{\lambda,\mu} : B_\lambda \to B_\mu$ as follows. This map sends each nonzero $u \in B_\lambda$ to the unique $v \in B_\mu$ such that u, v is a brace for λ, μ.

Let λ, μ denote adjacent locations in Δ_N.

We just defined an $\mathbb{F}$-linear map $\tilde{B}_{\lambda,\mu} : B_\lambda \to B_\mu$.

We now consider what happens when we compose the maps of this kind.

The maps $\tilde{B}_{\lambda,\mu}$, cont.

Lemma

Let λ, μ denote adjacent locations in Δ_N. Then the maps $\tilde{B}_{\lambda,\mu} : B_\lambda \to B_\mu$ and $\tilde{B}_{\mu,\lambda} : B_\mu \to B_\lambda$ are inverses.

The maps $\tilde{B}_{\lambda,\mu}$, cont.

Lemma

Let λ, μ, ν denote locations in Δ_N that form a white 3-clique. Then the composition around the clique:

$$B_\lambda \xrightarrow[\tilde{B}_{\lambda,\mu}]{} B_\mu \xrightarrow[\tilde{B}_{\mu,\nu}]{} B_\nu \xrightarrow[\tilde{B}_{\nu,\lambda}]{} B_\lambda$$

is equal to the identity map on B_λ.

The maps $\tilde{B}_{\lambda,\mu}$, cont.

Definition

Let λ, μ, ν denote locations in Δ_N that form a black 3-clique. Then the composition around the clique:

$$B_\lambda \xrightarrow[\tilde{B}_{\lambda,\mu}]{} B_\mu \xrightarrow[\tilde{B}_{\mu,\nu}]{} B_\nu \xrightarrow[\tilde{B}_{\nu,\lambda}]{} B_\lambda$$

is a nonzero scalar multiple of the identity map on B_λ. The scalar is called the **clockwise B-value** (resp. **c.clockwise B-value**) of the clique whenever the sequence λ, μ, ν runs clockwise (resp. c.clockwise) around the clique.

Clockwise and c.clockwise B-values

Lemma

For each black 3-clique in Δ_N, its clockwise B-value and c.clockwise B-value are reciprocals.

Definition

For each black 3-clique in Δ_N, by its B-**value** we mean the clockwise B-value.

We have now assigned a nonzero scalar value to each black 3-clique in Δ_N.

The value function $\hat{B}$

We define a function $\hat{B}$ on the set of black 3-cliques in Δ_N.

The function $\hat{B}$ sends each black 3-clique to its B-value.

We call $\hat{B}$ the **value function** for B.

The value function $\hat{B}$, cont.

It is convenient to view $\hat{B}$ as a function on Δ_{N-2}, as follows.

Pick $(r, s, t) \in \Delta_{N-2}$. Observe that the locations

$$(r, s+1, t+1), \quad (r+1, s, t+1), \quad (r+1, s+1, t)$$

are in Δ_N and form a black 3-clique.

The B-value of this 3-clique is equal to the image of (r, s, t) under $\hat{B}$.

Abstract value functions

We just defined the value function of a Billiard Array.

We will use these value functions to classify the Billiard Arrays up to isomorphism.

Definition

By a **value function** on Δ_N we mean a function $\psi : \Delta_N \to \mathbb{F}\backslash\{0\}$.

The classification of Billiard Arrays

We now classify the Billiard Arrays up to isomorphism.

Recall the Billiard Array B and its value function $\hat{B}$.

Theorem

The map $B \mapsto \hat{B}$ induces a bijection between the following two sets:

(i) *the isomorphism classes of Billiard Arrays over $\mathbb{F}$ that have diameter N;*

(ii) *the value functions on Δ_{N-2}.*

Our next goal is to describe what Billiard arrays have to do with 3-tuples of mutually opposite flags.

Until further notice let V denote a vector space over $\mathbb{F}$ with dimension $N + 1$.

Totally opposite flags

Definition

Suppose we are given three flags on V, denoted $\{U_i\}_{i=0}^N$, $\{U'_i\}_{i=0}^N$, $\{U''_i\}_{i=0}^N$. These flags are said to be **totally opposite** whenever $U_{N-r} \cap U'_{N-s} \cap U''_{N-t} = 0$ for all integers r, s, t $(0 \leq r, s, t \leq N)$ such that $r + s + t > N$.

Totally opposite vs mutually opposite

Given three flags on V, the totally opposite condition is somewhat stronger than the mutually opposite condition.

This is explained on the next slide.

Totally opposite vs mutually opposite

Lemma

Given three flags on V, denoted $\{U_i\}_{i=0}^N$, $\{U'_i\}_{i=0}^N$, $\{U''_i\}_{i=0}^N$. Then the following are equivalent:

(i) *the flags $\{U_i\}_{i=0}^N$, $\{U'_i\}_{i=0}^N$, $\{U''_i\}_{i=0}^N$ are totally opposite;*

(ii) *for $0 \le n \le N$ the sequences*

$$\{U_i\}_{i=0}^{N-n}, \qquad \{U_{N-n} \cap U'_{n+i}\}_{i=0}^{N-n}, \qquad \{U_{N-n} \cap U''_{n+i}\}_{i=0}^{N-n}$$

are mutually opposite flags on U_{N-n}.

We are going to show that the Billiard Arrays on V are in bijection with the 3-tuples of totally opposite flags on V.

To get started, we show how to get a Billiard Array on V from a 3-tuple of totally opposite flags on V.

Lemma

Suppose we are given three totally opposite flags on V, denoted $\{U_i\}_{i=0}^N$, $\{U'_i\}_{i=0}^N$, $\{U''_i\}_{i=0}^N$. For each location $\lambda = (r, s, t)$ in Δ_N define

$$B_\lambda = U_{N-r} \cap U'_{N-s} \cap U''_{N-t}.$$

Then the function B on Δ_N that sends $\lambda \mapsto B_\lambda$ is a Billiard Array on V.

Consider the following two sets:

(i) the 3-tuples of totally opposite flags on V;

(ii) the Billiard Arrays on V.

In the previous lemma we described a function from (i) to (ii).

Theorem

The above function is a bijection.

Conclusion

Our next goal is to use Billiard Arrays to explain what is special about the three standard flags for a finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$-module of type 1.

The main theorem

Theorem

Let V denote a finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$-module, with type 1 and dimension ≥ 2. Then:

(i) *the three standard flags on V are totally opposite;*

(ii) *for the corresponding Billiard Array on V, the value of each black 3-clique is a constant q^2.*

Summary

In this talk, we first considered a finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$-module V of type 1.

We defined three flags on V, called the standard flags.

We then introduced the notion of a Billiard Array on a vector space V.

We classified the Billiard Arrays up to isomorphism, using the notion of a value function.

We showed how the Billiard Arrays on V are in bijection with the 3-tuples of totally opposite flags on V.

We showed that for the above $U_q(\mathfrak{sl}_2)$-module V, the three standard flags are totally opposite, and for the corresponding Billiard Array the value function is constant, taking the value q^2.

Thank you for your attention!

THE END

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