Problem sheet

(1) Prove that there are no simple groups of order 312, 616, or 2014.

(2) Let $1 \longrightarrow A \xrightarrow{\mu} E \xrightarrow{\epsilon} G \longrightarrow 1$ be a group extension, where A is abelian and $G = \langle g \rangle$ cyclic of order n. Choose $x \in E$ with $x^\epsilon = q$, and let $a = x^n$. Define a transversal function $\tau : G \to E$ by $(g^i)^\tau = x^i$ for $0 \leq i < n$. Prove that the corresponding factor set $\phi : G \times G \to A$ is given by

$$(g^i, g^j)\phi = \begin{cases} 0 & : \quad i+j < n \\ a & : \quad i+j \geq n \end{cases}.$$

(3) Find all equivalence classes of extensions of $C_4 \times C_2$ by C_2. Which groups arise this way?

(4) Let G be a finite nilpotent group and N a non-trivial normal subgroup of G. Show the following:
  (a) $[N, G]$ is a proper subgroup of N.
  (b) Some maximal proper subgroup of N is normal in G.
  (c) Suppose that G is a p-group and M and N normal subgroups of G with $N < M$. Prove that there exists $K \lhd G$ such that $N \leq K < M$ and $|M : K| = p$.

(5) Suppose that $G = HN'$, where $H \leq G$ and $N \lhd G$. Prove that $G = H\gamma_i(N)$ for all i.