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CONSTRUCTION TECHNIQUES FOR GRAPH EMBEDDINGS

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Mathematicians have been trying to construct embeddings of specific graphs in surfaces since at least the 1890s. However, until the 1960s the construction techniques were usually fairly ad hoc, although some general ideas such as 'schemes of cyclic sequences' had emerged. This changed with the development of current graphs by Gustin and others in the 1960s, which provided a unified framework for many earlier constructions and played an important role in the proof of the Map Colour Theorem. Fifty years later we have a number of useful general tools for constructing embeddings of graphs. These lectures will survey tools of various kinds. We will look at algebraic methods such as current, voltage and transition graphs; surgical tools such as the diamond sum and adding handles or crosscaps around a vertex; lifting constructions due to Bouchet and his collaborators; and techniques that use objects from design theory, such as latin squares, to construct embeddings.

Note on presentation: These are lecture notes for a course that will survey a lot of material in a short amount of time, so the presentation is often informal and rigorous details are omitted.

The figures are taken from a number of different sources. Some are handdrawn, others are drawn using software packages. The author apologizes for the lack of consistency!

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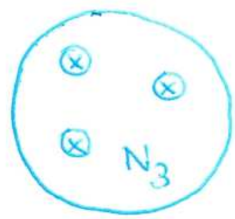
1. EMBEDDINGS OF GRAPHS

Surfaces

Definition: A *surface* is a 2-manifold without boundary. Examples: sphere, torus, projective plane, Klein bottle (all compact); plane, open Möbius strip (not compact).

Theorem, Classification of Surfaces: Every compact surface is homeomorphic to the sphere S_0, a sphere with $h \geq 1$ handles added S_h, or a sphere with $k \geq 1$ crosscaps added N_k.

Definition: *Adding a handle*: delete a disk, glue a punctured torus on to the boundary. *Adding a crosscap*: delete a disk, glue a punctured projective plane (i.e., a Möbius strip) on to the boundary.



Surfaces S_h, $h \geq 0$, are *orientable*: can define consistent clockwise orientation everywhere. Surfaces N_k, $k \geq 1$ are *nonorientable*: can travel in surface, maintaining locally consistent clockwise orientation, in such a way that orientation is reversed when you return to your starting point.

In an orientable surface all closed curves are 2-*sided*; nonorientable surfaces have 1-*sided* closed curves.

Question: What if add mixture of handles and crosscaps? Adding a crosscap and a handle is equivalent to adding three crosscaps. Consequently, if add $h \geq 0$ handles, $k \geq 1$ crosscaps, get N_{2h+k}.

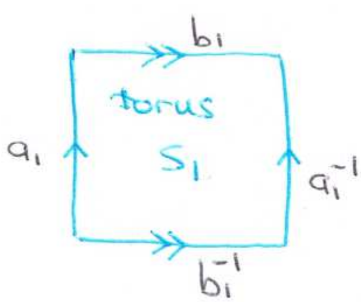
Definition: The *genus* of a surface is the number of added handles or crosscaps: genus of S_h is h, genus of N_k is k.

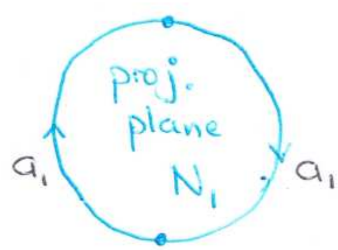
Convention: From now on 'surface' means 'compact surface' unless otherwise specified.

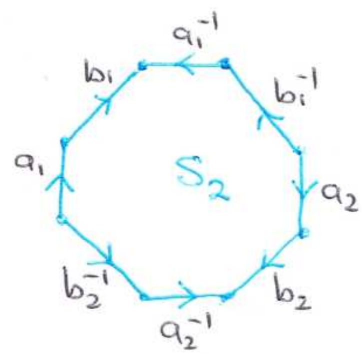
Representing surfaces

Polygon representation: Proof of classification theorem shows that every surface can be represented in a standard way as a polygon (possibly a 2-gon) with sides identified in pairs. Use inverse notation when sides identified in opposite directions.

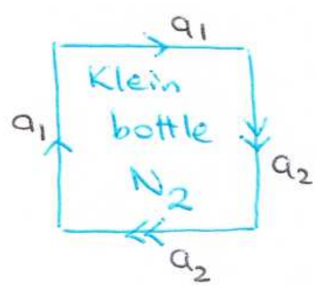
Sphere S_0: (aa^{-1})
S_h, $h \geq 1$: $(a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_h b_h a_h^{-1} b_h^{-1})$
N_k, $k \geq 1$: $(a_1 a_1 a_2 a_2 \ldots a_k a_k)$

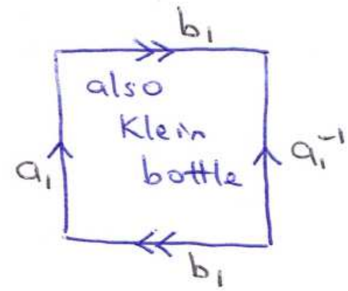




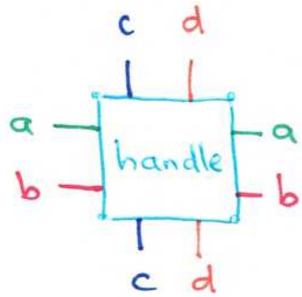


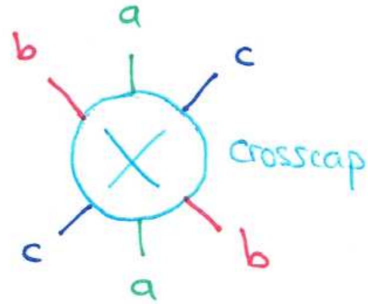
Surfaces can also be represented in other ways as polygons with identified sides, e.g. 'usual' representation of Klein bottle is not standard one.



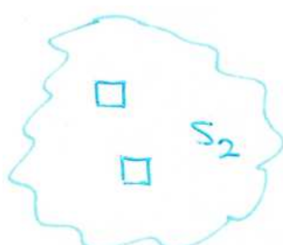


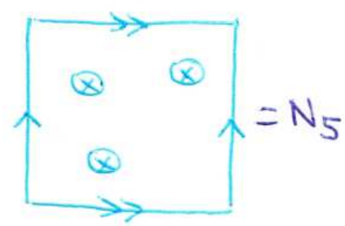
Planar representation with handle or crosscap gadgets: Can also represent surfaces in plane: think of sphere as plane with implicit point at infinity, then add handles or crosscaps which we treat as 'gadgets' allowing curves to cross in certain ways.





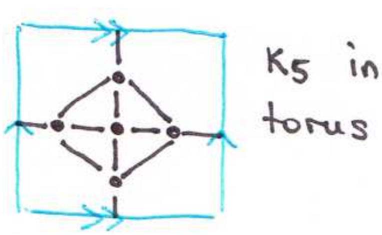
We can mix the above two representations: use polygon representation and then add handles or crosscaps.

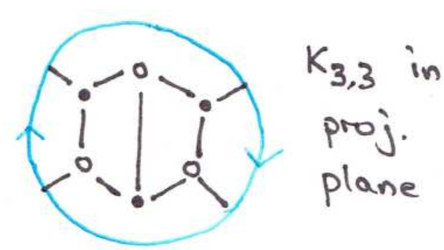




Graph embeddings

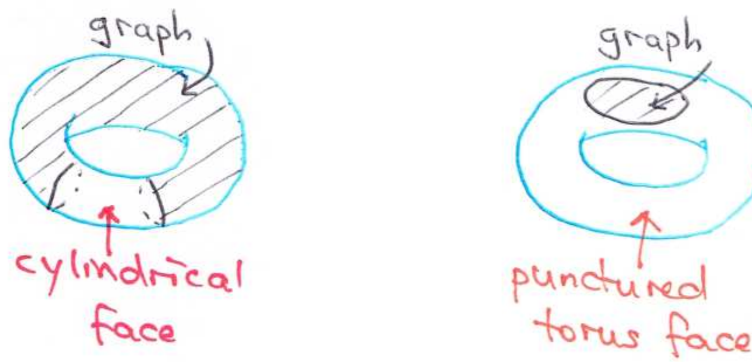
Definition: Loosely, an *embedding* Ψ *of graph* G *in surface* Σ, which we denote $\Psi : G \hookrightarrow \Sigma$, is a drawing of G in Σ with no crossing edges. Can make this rigorous, but concept should be clear.





Can represent embedding by drawing on either representation above (polygon with identified sides, or plane plus handle/crosscap gadgets), or on mixed representation. But if embedding nice, can represent in purely combinatorial ways or by simpler drawings.

Definition: Embedding of graph is *cellular* or open 2-cell or just 2-*cell* if every face is homeomorphic to an open disk.



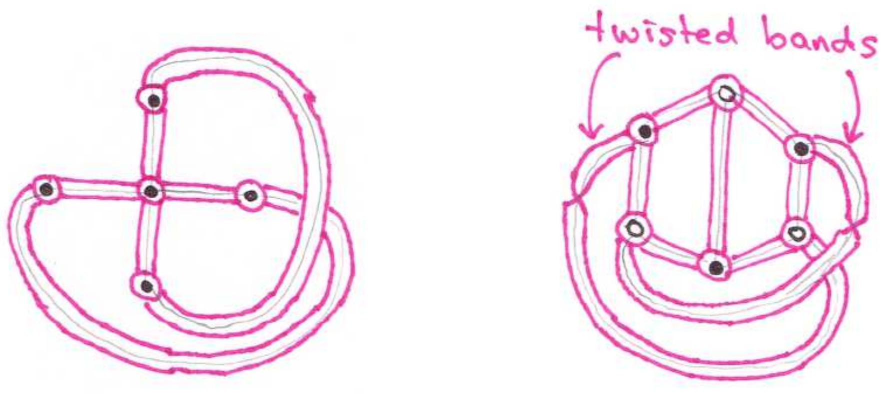
What prevents an embedding being 2-cell? Face has multiple boundary components, or face contains handles or crosscaps.

Even stronger definition: embedding is *closed* 2-*cell* if the closure of every face is homeomorphic to a closed disk. Equivalent to open 2-cell and boundary of every face is a cycle (not just a closed walk) in the graph. Closed 2-cell embeddings give *cycle double covers*. Closed 2-cell is usually a stronger property than we need or want.

Representation of 2-cell embeddings

Since all faces are open disks, just need to know how to glue faces onto graph.

Band decompositions or ribbon graphs: Take small disk around each vertex, small band (or strip) along each edge, throw rest of surface away. Get a 'fattened' version of graph. Can reconstruct entire surface by gluing a disk along each boundary component of resulting complex.



Rotation schemes: If our surface is orientable and we know a consistent global clockwise orientation, we can describe the embedding just by giving the clockwise order (*rotation*) of ends of edges at each vertex. This is a *pure rotation system*. Essentially known by Heffter in 1891, formalized by Edmonds in 1960.

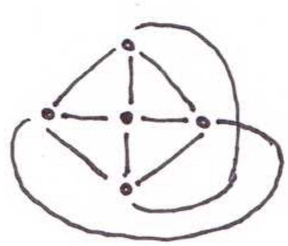
More general definition: If we do not know a consistent global clockwise orientation (always true if our surface is nonorientable, but surface could also be orientable) then we use a *local* clockwise orientation for each vertex to give the order of ends of edges. But then we need to say whether rotations at two ends of an edge match up.

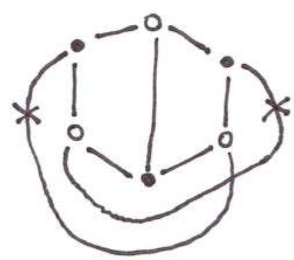
An edge is *type* 0 or *signature* 1 or *untwisted* if the local clockwise rotation of the vertex at one end can be followed along the edge and agrees with the local clockwise rotation of the vertex at the other end. Otherwise the edge is *type* 1 or *signature* -1 or *twisted*.

A *rotation scheme* in general consists of the orders of ends of edges around each vertex plus the type of each edge. This is a *purely combinatorial* description.

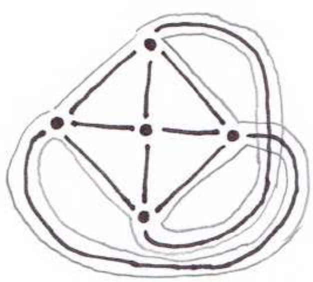
We can tell if a closed walk in a graph is 1-sided from this. A walk is 1-sided if and only if it contains an odd number of twisted (type 1) edges.

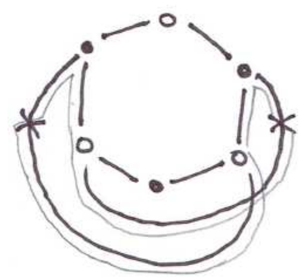
Rotation projections: However, it is convenient to represent a rotation scheme geometrically by a *rotation projection*. We just draw the graph in the plane, with edge crossings allowed, so that the clockwise order of ends of edges around each vertex agrees with the local clockwise orientation of the surface at that vertex. We indicate twisted (type 1) edges by putting an 'X' in the middle of them.





Face tracing for rotation projections: We can determine the face boundaries by following along the sides of edges, taking corners in the natural way, ignoring edge crossings, and switching sides in the middle of a twisted edge (at the 'X').





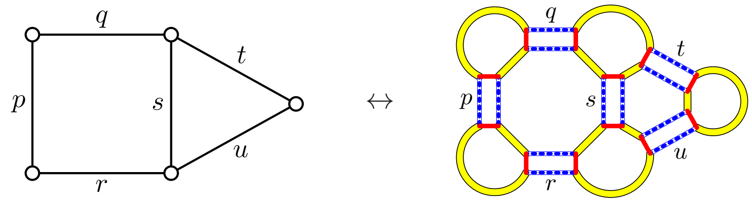
Orientability detection for rotation projections: The presence of twisted edges does not necessarily mean the embedding is nonorientable. Take spanning tree, start at root vertex, flip rotations so that all edges in tree become untwisted. Embedding orientable if and only if *all* edges now untwisted.

Gem representation: Due to Neil Robertson, 1971. Make band decomposition into 3-edge-coloured cubic graph:

Corner $\rightarrow$ vertex.
Vertex/face boundary $\rightarrow$ yellow edge.
Vertex/edge boundary $\rightarrow$ red edge.
Edge/face boundary $\rightarrow$ blue edge.

Embedded graphs $\leftrightarrow$ 3-edge-coloured graphs in which every red-blue cycle (edge) is a 4-cycle. Red-yellow cycles represent vertices, blue-yellow cycles represent faces.

Theory of gems developed extensively in book by Bonnington and Little [BL].



Facial walk description: Give collection of closed walks that cover every edge exactly twice. Can glue a disk along each such walk to get a surface provided have 'proper rotation' at each vertex, determined using 'rotation graph'.

Definition: The *rotation graph* at v has as vertices the ends of edges incident with v. Join two ends of edges if there is a face that passes through them consecutively. Rotation graph is *proper* if it consists of a single cycle.

Rotation graphs useful for building embeddings, basis of idea of transition graphs later.

Rotation graphs are useful for relative (partial) embeddings. E.g., rephrasing of theorem of Škoviera and Širáň, 1986: Given a graph G, a collection of closed walks using each edge at most twice can be completed to an embedding if and only if each rotation graph is a subgraph of a cycle (so is a spanning cycle, or is a collection of paths possibly including isolated vertices).

Embedding described by collection of facial walks is orientable if and only if can orient each walk so that every edge is used once in each direction.

Euler's formula

We have a fundamental counting relationship for graphs with 2-cell embeddings on surfaces.

Euler's formula: Suppose we have a 2-cell embedding of a connected graph G on a surface Σ, where G has v vertices, e edges, and the embedding has f faces. Then

$$v - e + f = \chi$$

where $\chi = \chi(\Sigma)$ is a constant that depends only on the surface; in particular,
$\chi(S_h) = 2 - 2h$ for $h \geq 0$ and
$\chi(N_k) = 2 - k$ for $k \geq 1$.

Definition: $\chi(\Sigma)$ is the *Euler characteristic* and can often be used to handle both orientable and nonorientable surfaces at the same time. But often convenient and more intuitive to have a nonnegative number with the same property. Define the *Euler genus* $\varepsilon(\Sigma)$ by
$\varepsilon(S_h) = 2h$ for $h \geq 0$ and
$\varepsilon(N_k) = k$ for $k \geq 1$
so that $\chi(\Sigma) = 2 - \varepsilon(\Sigma)$.

Example: K_5 on torus S_1: $v = 5$, $e = 10$, $f = 5$, $v - e + f = 5 - 10 + 5 = 0 = 2 - 2 \times 1$.

Important note: For Euler's formula to work, graph must be connected and embedding must be 2-cell. (There are more general versions that work if we relax these restrictions, but we need them for the basic formula above.)

Euler's formula and face degrees: Euler's formula gives an important implication involving the *degrees* of faces (lengths of facial walks) in an embedding. Since $\varepsilon = 2 - v + e - f$, for a minimum genus embedding of a given graph G (meaning v and e are fixed) we want to maximize f. Since the sum of the face degrees is $2e$, which is fixed, this means we want many faces of small degree. For a simple graph, we want triangular faces.

Based on considerations like this we can often find obvious lower bounds on the genus of embeddings of a given graph G. We then want to show that this lower bound can be achieved by constructing an embedding.

2. VOLTAGE GRAPHS

Note: Main reference for this section and next is Gross and Tucker's book [GT]. My notation and setup is similar to [GT], but not exactly the same.

Voltage graphs

Basic construction: Start with *base graph* G, orient each edge e arbitrarily to get directed edge e^+ in oriented graph $\vec{G}$, reverse of e^+ is e^-. (Oriented graph here refers to putting a direction on each edge, nothing to do with surfaces.)

Have *voltaage group* Γ (usually assumed to be finite), every edge assigned a weight or *voltage* $\alpha(e^+)$. Implicitly $\alpha(e^-) = \alpha(e)^{-1}$. Form *derived graph* G^α as follows:

$V(G^\alpha) = V(G) \times \Gamma$.

For each e^+ from u to v in $\vec{G}$ with $\alpha(e^+) = a$, add an (oriented) edge (e^+, g) in G^α from (u, g) to (v, ga) for every $g \in \Gamma$. (Reverse of (e^+, g) is (e^-, ga).) Edge directions can now be ignored.

Note: We multiply edge weights on right; could equally well define with edge weights multiplying on left.

At this point we have just constructed a *graph*, no embeddings yet.

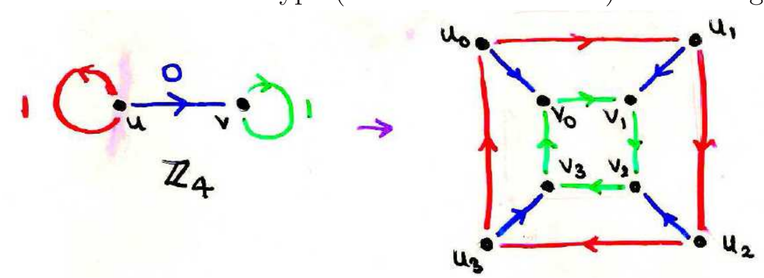
Remark: A *Cayley graph* is just a connected graph derived from a 1-vertex base voltage graph. Since G has only one vertex, vertices of G^α can be identified with elements of Γ.

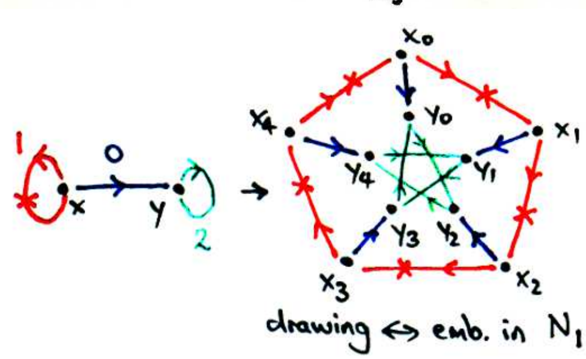
Embedded voltage graphs

Extension to embedded graphs: Suppose base graph G has 2-cell embedding Ψ in surface. Describe using rotation projection. Construct 2-cell embedding of derived graph with following additional rules:

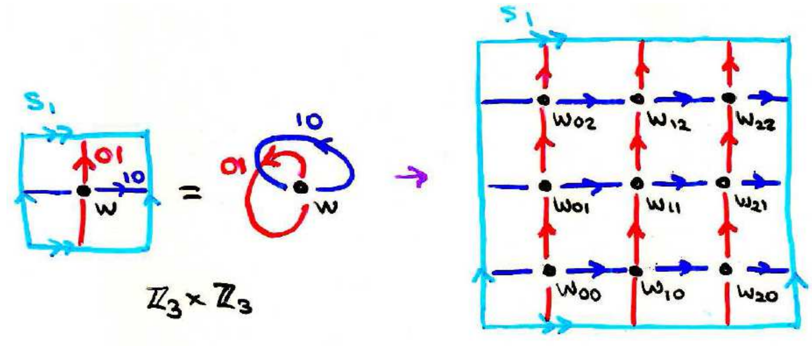
Around each vertex (v, g) of G^α the edges follow the order of their images in G (rotations are lifted).

Each edge in G^α has the same type (untwisted or twisted) as its image in G.





Can actually describe in more abstract terms without using rotation projection, but equivalent. Resulting *derived embedding* Ψ^α does not depend on specific rotation projection. Objects in Ψ or G are said to *lift* to corresponding objects in Ψ^α or G^α.



Lifting walks: Suppose uv-walk W in G corresponds to sequence of edges/reverse edges in $\vec{G}$ that is $f_1 f_2 \ldots f_d$. Say *net voltage* of W is $\alpha(W) = \alpha(f_1)\alpha(f_2)\ldots\alpha(f_d)$. If start at a vertex (u, g) in G^α and follow lifted walk $\tilde{W}$ in G^α, will end at $(v, g\alpha(W))$.

In particular, if W is a facial walk in G starting at u, will end at $(u, g\alpha(W))$. Will come back to original vertex (u, g) if repeat r times where r is the order of $\alpha(W)$ in Γ. Thus, each face of degree d in G becomes a face of degree dr in G^α where r is the order in Γ of the net voltage of the facial walk. Face length does not change exactly when net voltage is the identity (face satisfies *Kirchoff Voltage Law, KVL*).

Orientability: If original embedding of G is orientable, derived embedding will be orientable. If original embedding is nonorientable derived embedding could end up being orientable if all 1-sided walks lift to 2-sided walks.

Gross and Tucker [GT, 4.1.6] have algorithm based on reducing voltages in a spanning tree to the identity; won't discuss details.

But if voltage group has odd order all 1-sided closed walks have net voltage of odd order, must be repeated an odd number of times to close up in derived embedding, stay 1-sided, so embedding stays nonorientable.

More general voltage graphs

Permutation/group action voltage graphs: Gross and Tucker describe 'permutation voltage graphs' using permutation groups. Permutation groups are equivalent to group actions so can also describe that way. Suppose have right action of group Γ on set S: for each $s \in S$, $g \in \Gamma$ can form sg obeying natural rules.

Then given graph G with edges oriented and voltage $\alpha(e^+) \in \Gamma$ for each edge e, can form derived graph with

$V(G^\alpha) = V(G) \times S$.

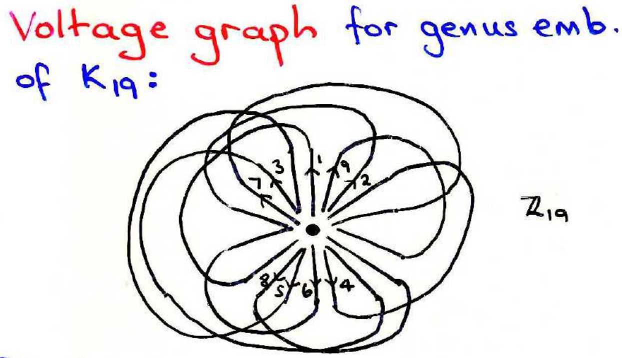
For each e^+ from u to v in G with $\alpha(e^+) = a$, add an edge (e^+, s) in G^α from (u, s) to (v, sa) for every $s \in \Gamma$. (Reverse of (e^+, s) is (e^-, sa).)

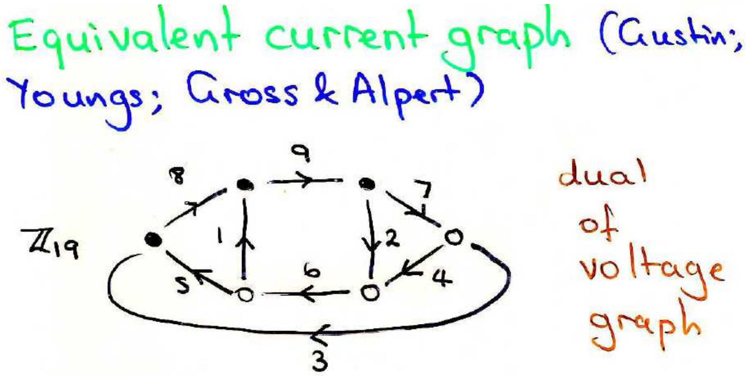
Can lift embedding of G to derived embedding of G^α in same way as for ordinary voltage graphs: lift vertex rotations and edge twists.

Final remark: Voltage graphs are straightforward to understand but may not be most convenient representation for particular applications. For very symmetric graphs a voltage graph representation of an embedding may have only one or two vertices and many edges, making it hard to keep track of where the edges go. So will look at alternative, current graphs, and then later another alternative, transition graphs.

3. CURRENT GRAPHS

Background: Current graphs were invented before voltage graphs, even though less intuitive. Used in proof of Map Colour Theorem, determination of minimum genus of complete graphs. Equivalent voltage graphs would have very few vertices, so it would be very hard to keep track of where the edges go.





Current graphs are duals of voltage graphs (so apply to *embedded* voltage graphs). Faces of current graph correspond to vertices in a voltage graph, and vice versa. However, tricky to deal with duals when have edge weights: need to turn them 90°, but which way? Hard to decide without globally consistent orientation (which never have in nonorientable case, and may not be given in orientable case).

See Gross and Tucker [GT] for general treatment. For simplicity we will restrict to current graphs given as rotation projections in plane.

Current graphs without twisted edges (hence orientable)

Basic construction: We are given oriented graph with weights or *currents* on edges, from *current group* Γ. For applications convenient to have two sorts of vertices (although only really need one sort):

solid $\bullet$ = clockwise vertices,

open $\circ$ = anticlockwise vertices.

Obtain derived embedding as follows:

Vertices of derived embedding have form (f, t) with f a face of the base graph and $t \in \Gamma$. To get faces of base graph with globally consistent rotations, trace faces in current graph in such a way that every edge is used once in each direction (trace all clockwise, or all anticlockwise). Order of edges along face f specify rotation around each vertex (f, t) in derived embedding.

Edges of derived embedding have form $(f, t)(g, ta)$, where f and g are faces of base graph meeting along an edge of current a; decide which way current applies based on rules below.

Faces of derived embedding come from vertices of base graph. For each vertex multiply currents of incident edges together in direction of vertex to get net current. Order r of net current in current group specifies how many times that sequence of edges is repeated to give a face of the derived graph, so vertex of degree d yields face of degree dr.

If net current of vertex is the identity, say vertex obeys the *Kirchoff Current Law, KCL*. Then vertex of degree d yields face of degree d.

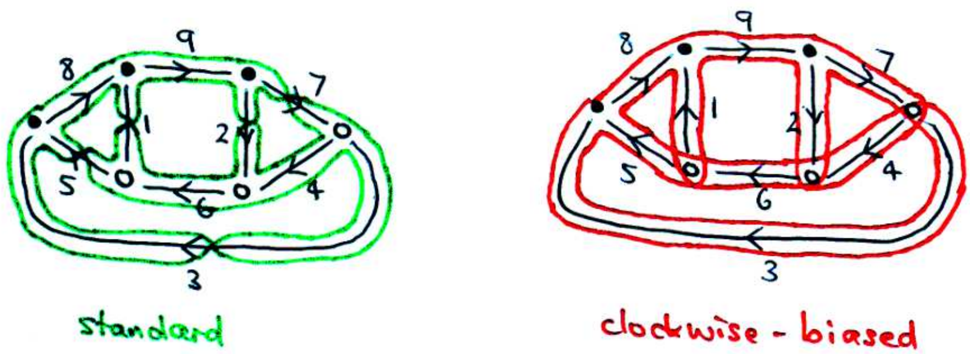
Standard tracing algorithm:

at each vertex follow natural rotation;

if an edge has vertices of different directions at its ends, cross over in the middle;

if we are leaving a clockwise vertex on an edge with face f on left, face g on right, current is a, then in derived graph (f, t) is joined to (g, ta) for each $t \in \Gamma$ (current acts 90° clockwise because vertex is clockwise); for an anticlockwise vertex swap left $\leftrightarrow$ right (current acts 90° anticlockwise).

There are alternative ways to trace faces that are more convenient in some ways.

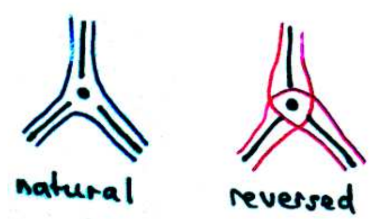


Clockwise-biased tracing algorithm:

at clockwise vertices follow natural rotation;

at anticlockwise vertices follow reversed rotation;

if going along an edge with face f on left, face g on right, current is a, then in derived graph (f, t) is joined to (g, ta) for each $t \in \Gamma$ (currents *always* act 90° clockwise).



Advantage is that we don't have to worry about vertex rotations until we are actually at vertex. Also less complicated when have to deal with twisted edges, later.

Also have *anticlockwise-biased tracing algorithm*: swap clockwise $\leftrightarrow$ anticlockwise, left $\leftrightarrow$ right. Can choose whether to use clockwise-biased or anticlockwise-biased algorithm depending on whether more clockwise or anticlockwise vertices.

All tracing algorithms give same result. In each case, resulting list of edges, destination faces, and (outgoing) currents is called *log* of face.

In above examples, only one face, and edges uniquely identified by current, so can just write log by listing currents:

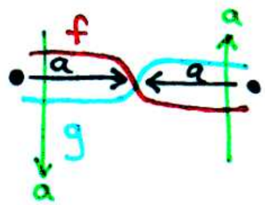
$8 \quad 9 \quad 7 \quad 4 \quad -2 \quad -9 \quad -1 \quad 5 \quad -3 \quad -7 \quad 2 \quad 6 \quad 1 \quad -8 \quad -5 \quad -6 \quad -4 \quad 3$

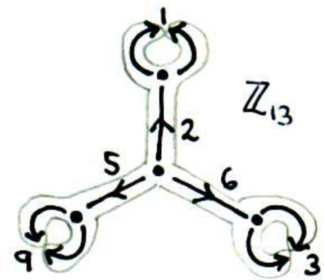
Current graphs with twisted edges

Principle for handling twisted edges: Twisted edges reverse whether we cross over in the middle of the edge or not. To maintain consistency of rules about the way currents act, when we go through a twist on an edge, current must reverse. Current reverses in middle of edge, so twisted edges have same current going in opposite directions on opposite ends.

Modifying standard tracing algorithm: When traverse a twisted edge, cross over in middle if vertices at ends have same direction, do not cross if vertices have opposite directions.

Modifying clockwise- or anticlockwise-biased tracing algorithms: Always cross over in middle of a twisted edge.





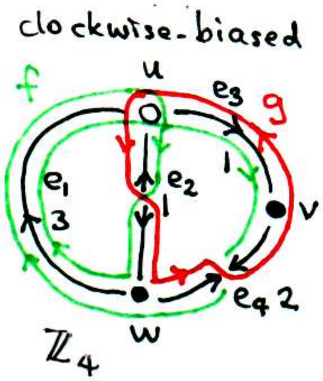
Dealing with lack of global orientation in derived graph: Unless base embedding is actually orientable, cannot trace faces in consistent way, using each edge once in each direction. Give each vertex (f, t) local (clockwise) rotation based on order edges encountered when tracing face f. In derived embedding suppose edge $e' = (f, t)(g, ta)$ is derived from e with f, g on either side. Then e' is twisted if both f and g trace e in the same direction, untwisted if they trace it in opposite directions.

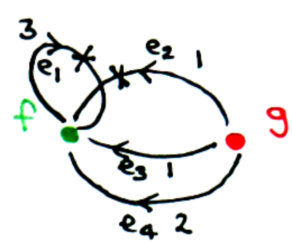
Example: For $\mathbb{Z}_{13}$ current graph above, again just one face, edges uniquely identified by currents. Log of face is

$2 \quad 1^* \quad -1^* \quad -2 \quad 6 \quad 3^* \quad -3^* \quad -6 \quad 5 \quad 9^* \quad -9^* \quad -5$

where * denotes twisted edge in derived embedding.

Even with twisted edges derived embedding may be orientable (just as for nonorientable voltage graphs). Will always be nonorientable if base graph is actually nonorientable and current group has odd order.





Example: See figure above. Face logs need to show edge, current applied, face representing other end in derived embedding, and also whether edges are twisted (denoted by *).

$$
\begin{array}{cccccccccc}
f: & e_1^* & e_2^* & e_1^* & e_3 & e_4 & \qquad g: & e_4 & e_3 & e_2^* \\
 & 3 & -1 & 3 & -1 & -2 & & 2 & 1 & 1 \\
 & f & g & f & g & g & & f & f & f
\end{array}
$$

We show the equivalent voltage graph.

Final remarks on current graphs

Map Colour Theorem: Current graphs were used heavily to determine the minimum genus of the complete graph K_n, generally by finding triangular embeddings. This often meant using current graphs with one face ('index one') and with most vertices of degree 3 and net (additive) current 0 (satisfying KCL). Ringel's book on this [Ri] uses current graphs very heavily; presentation sometimes disagrees with modern conventions.

Using both voltages and currents together: Won't go into details, but can use voltages and currents simultaneously on an embedding by applying voltages to edges of gem representation, in such a way that net voltage of each red-blue cycle (corresponding to an edge) is the identity. Equivalent to a construction by Dan Archdeacon [Ar92] that puts voltages on edges of *medial graph*.

Exercise: Suppose we have a voltage graph G and we take a vertex v and a constant $g \in \Gamma$. If we right multiply all voltages on edges into v by g, and left multiply all voltages on edges out of v by g^{-1}, then the derived graph stays unchanged, except that vertex (v, h) is now labelled vertex (v, hg) for each $h \in \Gamma$.

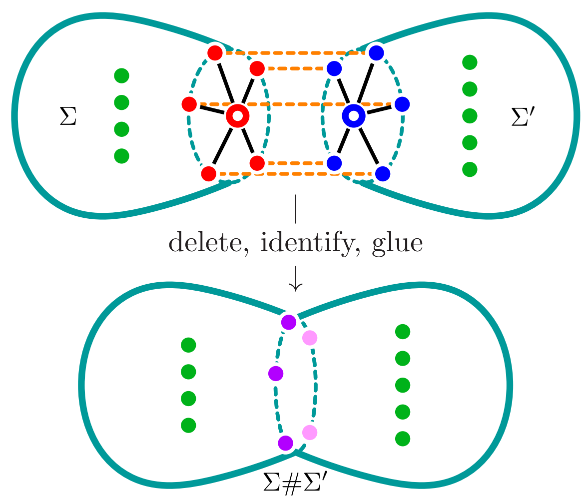
Prove that if we have a gem with assigned voltages, such that the net voltage around every red-blue cycle is the identity, then we can modify the voltages as in the previous paragraph, so that all red and blue edges have identity voltage. [Then all nontrivial voltages are on yellow edges; this is the main step in proving that assigning voltages to gems is effectively the same as Archdeacon's assignment of voltages to the medial graph.]

4. BOUCHET'S DIAMOND SUM

Definition: To take the *diamond sum* of two graphs G and G' we take vertices v in G and v' in G', so that $\deg_G(v) = \deg_{G'}(v')$, delete the two vertices, and identify their neighbours together. We denote this as $G \diamond G'$ (where v, v', and the particular identification of their neighbours are understood to be known).

We can extend this to two embeddings Ψ of G and Ψ' of G': when we delete v and v' we cut along a curve through their neighbours, and we glue the surfaces together (to get *connected sum* surface $\Sigma\#\Sigma'$). The neighbours must be identified in rotation order. We denote this as $\Psi \diamond \Psi'$.

Note that if Ψ, Ψ' have Euler genus ε, ε' respectively, then $\Psi \diamond \Psi'$ has Euler genus $\varepsilon + \varepsilon'$. So if both embeddings are orientable, or both are nonorientable, we can just add the genera of the surfaces.



History: Used by Bouchet [Bo78a] in dual form for new proof of minimum genus of $K_{m,n}$, 1978. Primal form used by Mohar, Parsons and Pisanski [MPP85], and Magajna, Mohar and Pisanski [MMP86], mid-1980s. Mohar and Thomassen [MT] give primal version of Bouchet's proof in their book and use diamond notation, hence name 'diamond sum'. General form stated by Kawarabayashi, Stephens and Zha [KSZ04].

Theorem: The minimum genus of an orientable genus embedding of $K_{m,n}$ is $\mathbf{g}(K_{m,n}) = \lceil (m-2)(n-2)/4 \rceil$.

Proof: This was first proved by Ringel, 1965. But we will give a proof based on the diamond sum, following the one in Mohar and Thomassen [MT]. Will use straightforward primal arguments instead of Bouchet's dual arguments.

Euler's formula and the fact that face degrees must be at least 4 (since graph is simple and bipartite) gives a lower bound of $f_0(m,n) = (m-2)(n-2)/4$ on genus, which is achieved if we have a quadrangular embedding (all facial walks are 4-cycles). Since genus is integral, can round up: $\mathbf{g}(K_{m,n}) \geq f(m,n) = \lceil f_0(m,n) \rceil = \lceil (m-2)(n-2)/4 \rceil$.

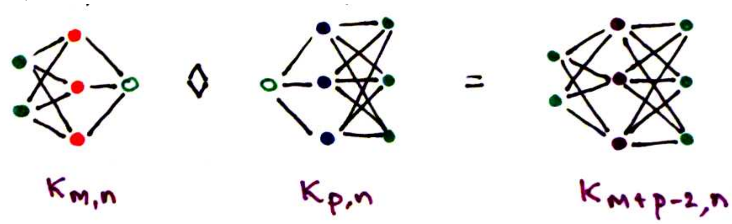
$F(m,n)$ is the statement that $\mathbf{g}(K_{m,n} = f(m,n)$. We prove it for $m,n \geq 2$ by induction on $m+n$ by constructing an embedding. Note that $F(m,n) \Leftrightarrow F(n,m)$. True if $m=2$ or $n=2$ so suppose $m,n \geq 3$.

Claim D: If $F(m,n)$ and $F(p,n)$ hold and at least one of $f_0(m,n)$ and $f_0(p,n)$ is integral, then $F(m+p-2,n)$ holds.

Proof: Take the diamond sum of minimum genus embeddings of $K_{m,n}$ and $K_{p,n}$, deleting a vertex in the first part of each bipartition. The resulting graph is $K_{m+p-2,n}$ with an embedding of genus

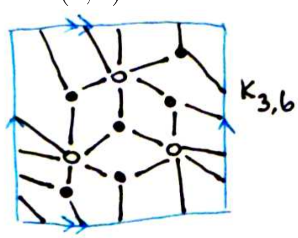
$$\begin{aligned} f(m,n) + f(p,n) &= \lceil f_0(m,n) \rceil + \lceil f_0(p,n) \rceil \\ &= \lceil f_0(m,n) + f_0(p,n) \rceil \quad \text{as long as one of } f_0(m,n),\ f_0(p,n) \text{ is integral} \\ &= \lceil \frac{(m-2)(n-2)}{4} + \frac{(p-2)(n-2)}{4} \rceil = \lceil \frac{(m+p-4)(n-2)}{4} \rceil \\ &= \lceil f_0(m+p-2,n) \rceil = f(m+p-2,n) \end{aligned}$$

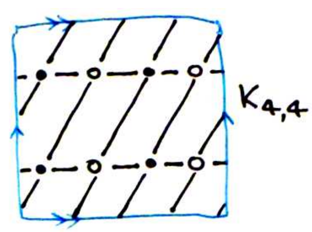
and so $F(m+p-2,n)$ holds.



Claim B: $F(3,6)$ and $F(4,4)$ hold.

Proof:





Claim S: $F(m,6)$ holds for all m.

Proof: By repeated diamond sums with $K_{3,6}$ we can build up $K_{3,6} \to K4,6 \to K5,6 \to \ldots$, and since $f_0(3,6) = 1$ is integral the result follows from Claim D.

Claim B$^+$: $F(m,n)$ holds if $m,n \leq 6$.

Proof: Use Claim S if $m=6$ or $n=6$. Claim B covers $F(4,4)$, and also $F(3,6)$ from which we also get $F(3,3)$, $F(3,4)$ and $F(3,6)$. We get $F(4,5)$ from $F(4,6)$, and $F(5,5)$ from $F(5,6)$.

Now we just use induction. Without loss of generality $m \leq n$ and $n \geq 7$. Now $F(m,n-4)$ and $F(m,6)$ give $F(m,n)$ by Claim D. ∎

Nonorientable genus of $K_{m,n}$: In a similar way can prove that $K_{m,n}$ has nonorientable genus $\widetilde{\mathbf{g}}(K_{m,n)} = \lceil (m-2)(n-2)/2 \rceil$.

Minimum genus of complete tripartite graphs

Use of diamond sums suggested by Kawarabayashi, Stephens and Zha [KSZ04]. Used by Ellingham, Stephens and Zha [ESZ06] (together with transition graphs and surgical techniques) to find nonorientable genus of all complete tripartite graphs.

Lower bound from Euler's formulas, conjectured to give actual genus: assume $\ell \geq m \geq n$:

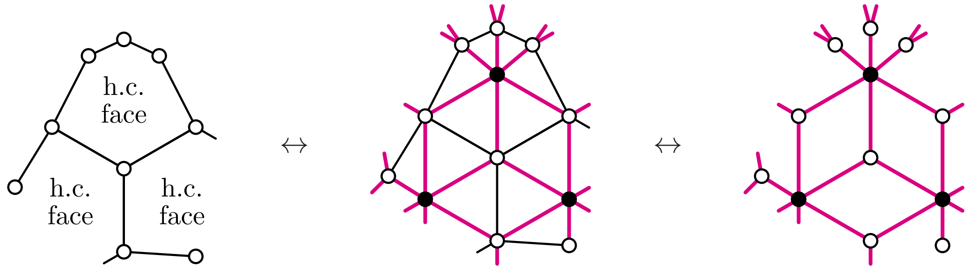
$\mathbf{g}(K_{\ell,m,n}) \geq \lceil(\ell-2)(m+n-2)/4\rceil$,
$\widetilde{\mathbf{g}}(K_{\ell,m,n}) \geq \lceil(\ell-2)(m+n-2)/2\rceil$.

Note that lower bound is just genus of $K_{\ell,m+n}$. So if this is really the genus, a minimum genus embedding of $K_{\ell,m,n}$ just consists of a minimum genus embedding of $K_{\ell,m+n}$ with the edges of a $K_{m,n}$ inserted into the faces without changing the surface.

So diamond sum works in a way similar to complete bipartite graphs, but we have extra edges of a $K_{m,n}$ on one side of the diamond sum. Specifically, we can take diamond sum of $K_{\ell,m,n}$ with $K_{p,m+n}$, deleting vertex in first part of partition of each graph. Result is $K_{\ell+p-2,m,n}$. Means that we can start with embeddings of $K_{\ell,m,n}$ for only a small number of values of ℓ close to m (at worst $m, m+1$ in nonorientable case or $m, m+1, m+2, m+3$ in orientable case: stop at first value where no rounding occurs in formula above) and then get all other values of ℓ by diamond sum.

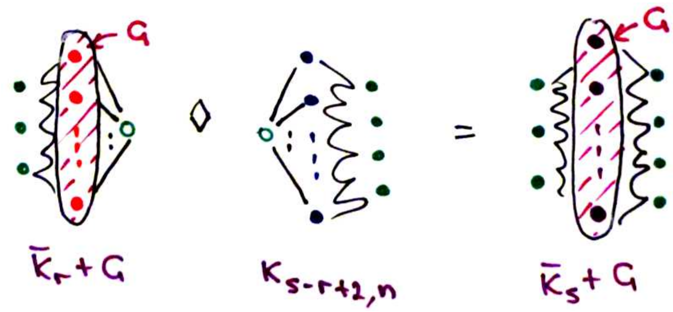
Genus of families of graphs from hamilton cycle embeddings

The situation with complete tripartite graphs suggested looking at graphs that look like complete bipartite graphs with some extra edges added on one side of the bipartition. Turns out to be related to embeddings where all facial walks are hamilton cycles.



Hamilton cycle embedding of some r-*regular* n-vertex G
$\leftrightarrow$ Triangular (hence min. genus) embedding of *join* $\overline{K_r} + G$
$\leftrightarrow$ Quadrangular (hence min. genus) embedding of $K_{r,n}$.

So the middle step here is a complete bipartite graph with edges added on one side, and the last step tells us that we added edges to a minimum genus embedding of $K_{r,n}$. So we can now proceed as follows:



Hamilton cycle embedding of r-regular n-vertex G
$\updownarrow$

Triangular (hence minimum genus) embedding of join $\overline{K_r} + G$

$|$

because contains min. genus emb. of $K_{r,n}$

$\downarrow$

Minimum genus embedding of join $\overline{K_r} + H$ for any spanning subgraph H of G

$|$

diamond sum with min. genus embedding of $K_{n,s-r+2}$

$\downarrow$

Guaranteed minimum genus embedding of $\overline{K_s} + H$ for all $s \geq r$ and spanning $H \subseteq G$.

Exercise: Prove that the nonorientable genus of $K_{m,n}$ is $\lceil (m-2)(n-2)/2 \rceil$ for $m, n \geq 3$, given that this is known to be a lower bound on the genus. First find an embedding of $K_{3,4}$ in the projective plane N_1. Then use the diamond sum for induction.

5. TRANSITION GRAPHS

Comment on the name: In retrospect 'transition graph' is not a great name. Should really be called 'global rotation graphs' or something like that: name comes from fact that edges in rotation graph represent 'transitions' between two edges as we pass through a vertex.

General idea: Given an embedded voltage graph, take rotation graph around each vertex R_v. Now for each edge e from u to v identify the vertex of R_u corresponding to an end of e with the vertex of R_v corresponding to the other end of e. Result is actually medial graph of voltage graph. Add some information corresponding to embedding of medial graph, edge twists, voltages.

Will not give formal definition. If desired, see [ESZ06].

Scope and usefulness: This is a general construction, equivalent to embedded voltage graphs (or to current graphs).

We saw that current graphs were more convenient than voltage graphs for finding triangular embeddings of complete graphs. Similarly, transition graphs are more convenient for embeddings of regular complete bipartite graphs $K_{m,m}$ with control over face sizes (usually want faces to be either 4-cycles or bamilton cycles). Play a key role in determining genus of complete tripartite graphs.

Controlled embeddings of $K_{m,m}$

Motivation: For complete tripartite graphs of form $K_{m,m,n}$, may get min. genus embedding from embedding of $K_{m,m}$ with

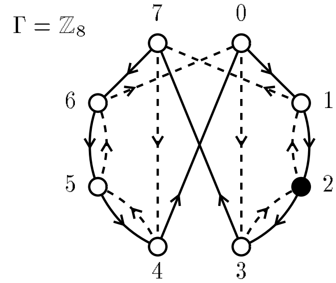
n hamilton cycle faces,

all other faces 4-cycles.

Can then add n new vertices in the hamilton cycle faces.

For joins of edgeless and complete graphs of form $\overline{K_m} + K_m$, may get min. genus embedding from embedding of $K_{m,m}$ with room in faces to add edges of a K_m.

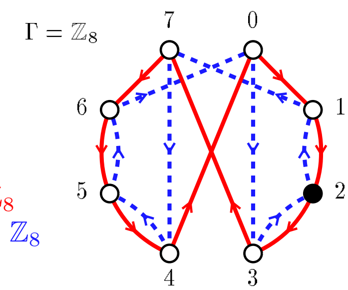
Structure of a transition graph: Construction has
group Γ, directed graph D,
vertices (not edges) labelled by voltages in Γ,
edges partitioned into directed cycles,
each vertex traversed exactly twice by directed cycles,
vertices (not edges) may have twist (solid vertex $\bullet$).



For embeddings of $K_{m,m}$ generally have
group $\Gamma = \mathbb{Z}_m$,
exactly two directed cycles (solid, dashed).

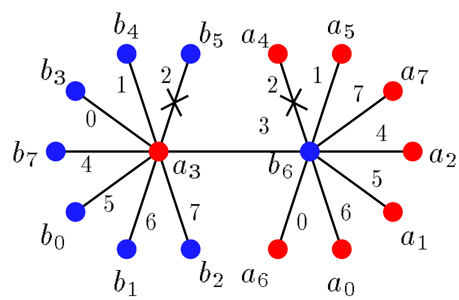
Deriving the embedding:
Directed cycle $\to$ vertices indexed by Γ,
vertex $\to$ class of edges with given "slope",
twisted vertex $\to$ twisted edges,
directed cycles show rotations.

Example:

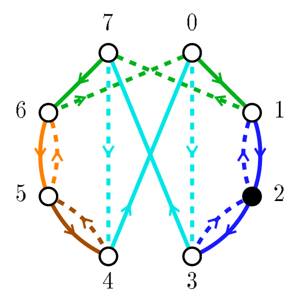


solid $\to a_i$, $i \in \mathbb{Z}_8$
dashed $\to b_i$, $i \in \mathbb{Z}_8$

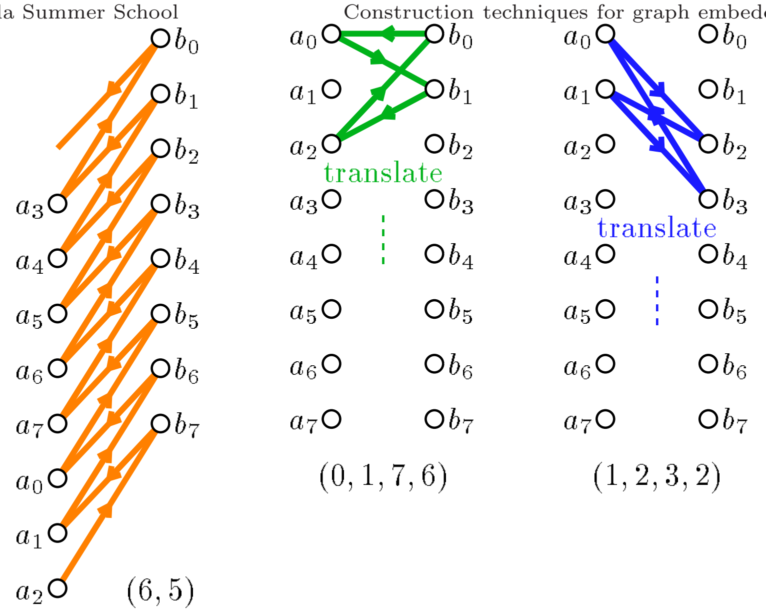
gives as part of derived embedding



Tracing faces:
Follow edges in transition graph,
switching directed cycles at each vertex,
at twisted vertex also switch directions.



Results: $(0, 1, 7, 6)$, $(1, 2, 3, 2)$, $(4, 0, 3, 7)$, $(6, 5)$, $(5, 4)$ – give consecutive *slopes* (voltages) of edges in faces.

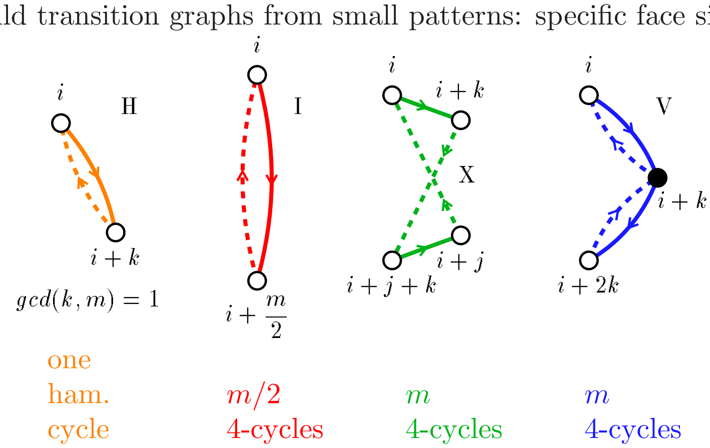


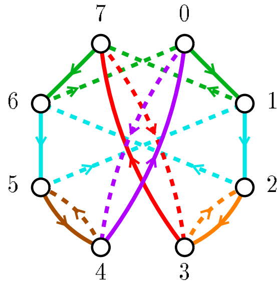
Advantages of transition graphs

- Can be built up from small patterns representing groups of faces of a particular size (H, I, V, X, S, $\ldots$).
- Can be used to build whole families of embeddings at once, by making substitutions involving small patterns ($2H \leftrightarrow V$, $4H \leftrightarrow 2X$).
- Can be used to build relative (partial) embeddings, then complete with "gadgets" (non-algebraic constructions), when completely algebraic construction is impossible.
- Allow very precise control of emb. structure:
  set up places to add edges;
  set up ways to extend embedding using vertex duplication or special diamond sums.

Building up from small patterns

Easy to build transition graphs from small patterns: specific face sizes.





Embedding of $K_{8,8}$:

X I

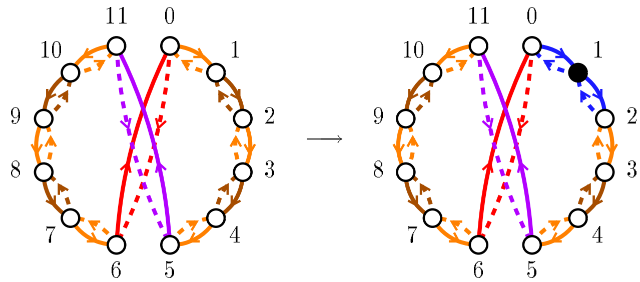
$8+8+4+4 = 24$ 4-cycle faces

H

$1 + 1 = 2$ ham. cycle faces

$\rightarrow$ min. genus embedding of $K_{8,8,2}$ on S_{12}.

Building families of embeddings



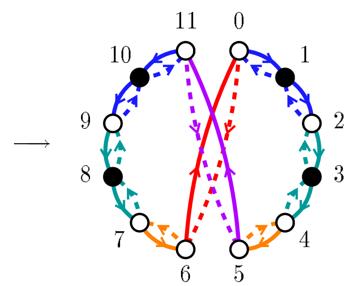
- Switch $2H \rightarrow V$ (nonorientable):

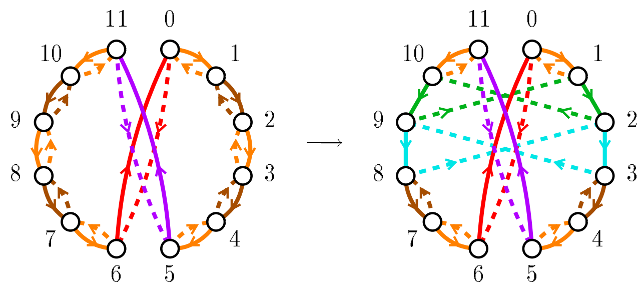
  $K_{12,12}$ with 10 ham. cycle faces
  $\rightarrow$ Ori. min. genus emb. of $K_{12,12,10}$

  which is modified to give

  $K_{12,12}$ with 8, 6, 4, 2 ham. cycle faces
  $\rightarrow$ Nonori. min. genus emb. of $K_{12,12,8/6/4/2}$

$\vdots$



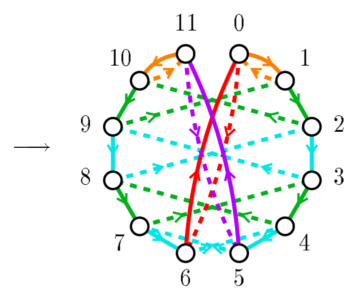


- Switch $4H \rightarrow 2X$:

  $K_{12,12}$ with 10 ham. cycle faces
  $\rightarrow$ Ori. min. genus emb. of $K_{12,12,10}$

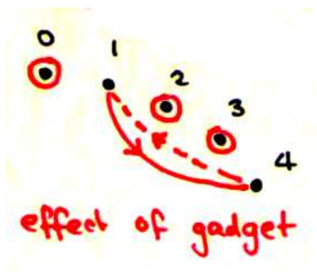
  which is modified to give

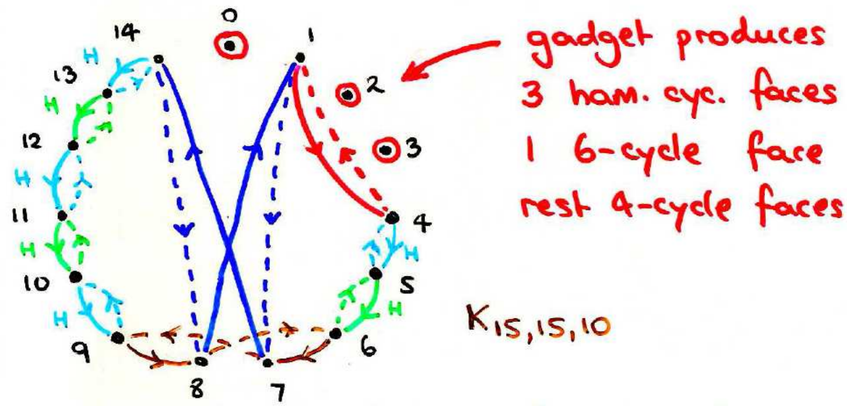
  $K_{12,12}$ with 6, 2 ham. cycle faces
  $\rightarrow$ Ori. min. genus emb. of $K_{12,12,6/2}$



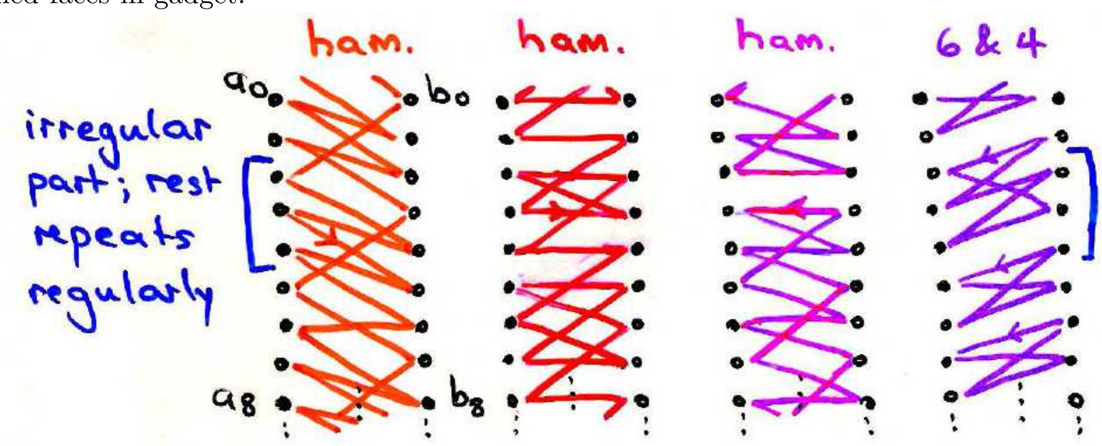
Gadgets

Sometimes there is no purely algebraic way to construct an embedding of $K_{m,m,n}$ using a transition graph. Instead use a *partial transition graph* together with a *gadget*, a set of faces not perfectly symmetric under the action of $\mathbb{Z}_m$, but which easily generalizes.





Detailed faces in gadget:



Special transition graphs (adding edges)

Can also do other things with transition graphs. For example, by controlling the lengths of edges (length of $i \to j$ is $j-i$) we can control which vertices share faces. If we get edges of one type (solid or dashed) with all possible lengths, means vertices in one class share a face with every other vertex in the same class, so can add a complete graph on that side of the bipartition. Used for example to construct orientable minimum genus embeddings of $\overline{K_n} + K_n$ for even n.

Exercise: Find a transition graph that generates an embedding of $K_{14,14}$ with twelve hamilton cycle faces and all other faces being 4-cycles.

Now repeat for eleven hamilton cycle faces.

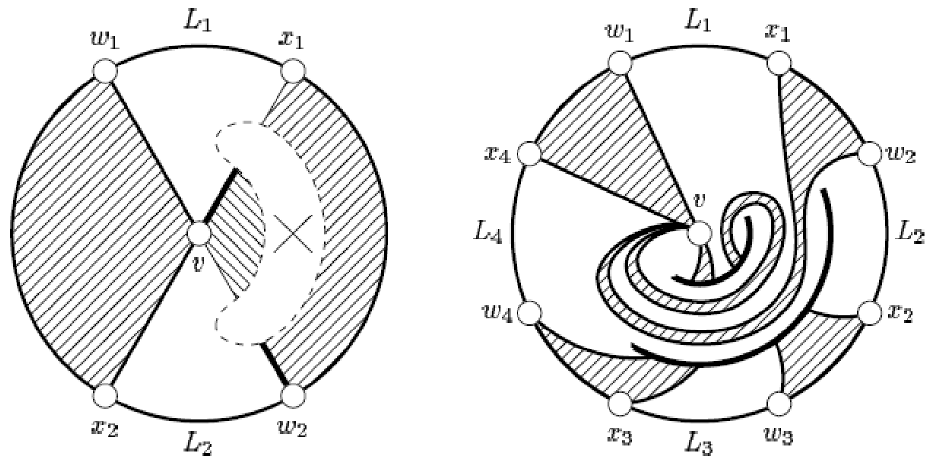
(These allow us to get minimum genus embeddings of $K_{14,14,12}$ and $K_{14,14,11}$.)

6. SURGERY

Surgery (cutting and pasting) can be used in many ways. Two very typical ways are for local modification of embeddings and for recursive constructions. Will give illustrations for each.

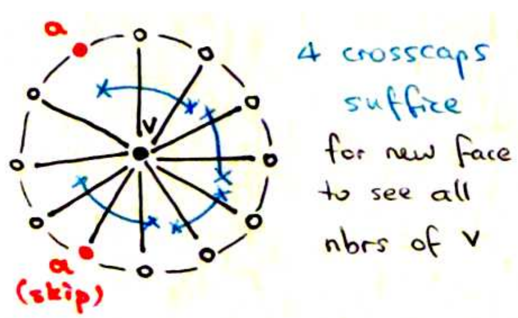
Local modifications

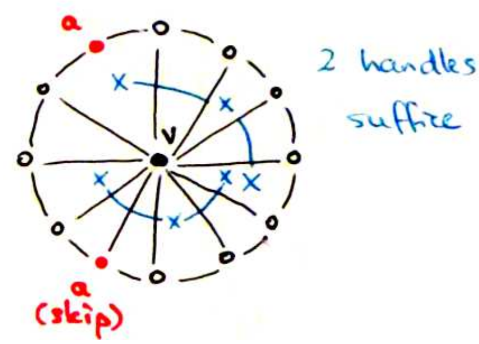
Merging faces around a vertex: Can use a single crosscap to merge two faces around the same vertex into a single face. Similarly, can use a handle to merge three faces around the same vertex into a single face.



By repeating this process we can merge enough faces around a given vertex v into a single face so that we can add into the new face a new vertex v' that is adjacent to all neighbours of v. We call this *duplicating a vertex*. See [ESZ06]; similar ideas also used by other people.

The problem is often that we wish (when constructing minimum genus embeddings) to use only a certain number of crosscaps or handles. We may have to be careful and creative in how we place the crosscaps or handles.

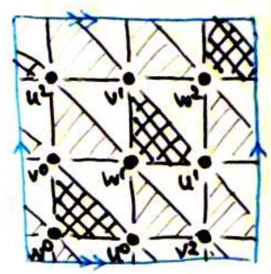




Recursive constructions

'Tripling' for triangulations of complete graphs: Grannell, Griggs and Širáň [GGS98] use 2-face-colourable triangulation of K_n to construct 2-face-colourable triangulation of K_{3n-2}. (Face colouring is important.)

- Take triangulation of K_n, cut out one vertex z, now have K_{n-1} on surface with boundary S.
- Take three copies of S: S^0, S^1, S^2, where v^i on S^i corresponds to v on S, etc.

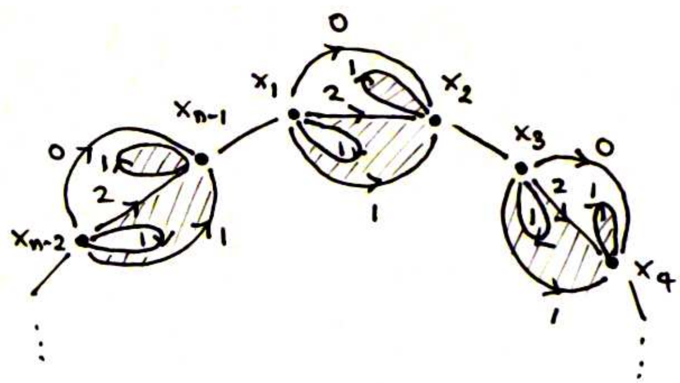


- For each white triangle $t = (uvw)$ cut out t^0, t^1, t^2 and glue on 2-face-colourable toroidal embedding of $K_{3,3,3}$ with vertex classes $\{u^0, u^1, u^2\}$, $\{v^0, v^1, v^2\}$, $\{w^0, w^1, w^2\}$ which has three black triangles $(u^i v^i w^i)$ deleted. Gives all edges of K_{3n-3} except those $x^i y^j$ where $i \neq j$ and xy incident with boundary and black triangle (then no white triangle containing that edge), and edges of form $x^i x^j$ where $i \neq j$.

- Now suppose boundary is $(x_1 x_2 \dots x_{n-1})$ (where $n-1$ is even) where x_1x_2, x_3x_4, … are incident with only black triangles. Construct derived embedding from $\mathbb{Z}_3$-voltage graph shown: contains

  cycles $(x_1^i x_2^i \dots x_{n-1}^i)$ to glue on to boundaries of S^0, S^1, S^2,

  assuming $3 \not| n-1$, hamilton cycle $(x_1^0 x_2^1 x_3^1 x_4^2 \dots x_{n-1}^0)$ in which to add extra vertex, and all missing edges;



This construction is important: by varying the way the $K_{3,3,3}$ embeddings are glued on, was first construction of large number (c^{n^2}) of nonisomorphic triangular embeddings of given complete graphs K_n [BGGS00].

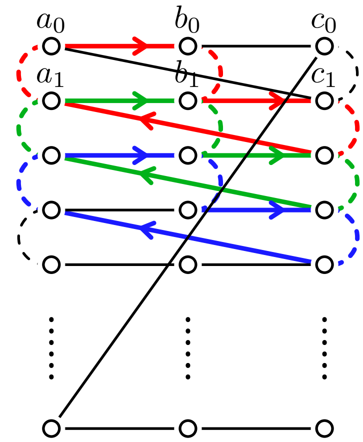
'Doubling' and 'tripling' for hamilton cycle embeddings of complete graphs: Due to Ellingham and Stephens [ES09]/Ellingham and Schroeder [ES14b] Use hamilton cycle embedding of regular complete bipartite/tripartite graph (known) to glue together hamilton cycle embeddings of K_n to get hamilton cycle embedding of K_{2n-2} or K_{3n-3}.

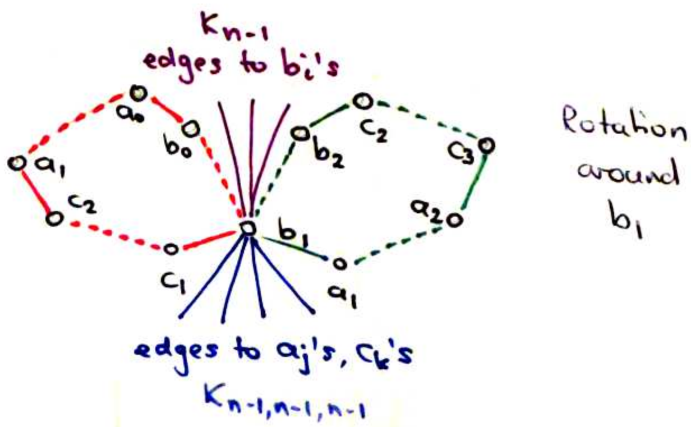
For 'tripling', glue together:

(a) three hamilton cycle embeddings of K_n, each with one vertex deleted, and

(b) one hamilton cycle embedding of $K_{n-1,n-1,n-1}$ with at least one *abc*-pattern face (which we remove).

Result is hamilton cycle embedding of K_{3n-3}. Rotation around b_1 shown to see how it works.





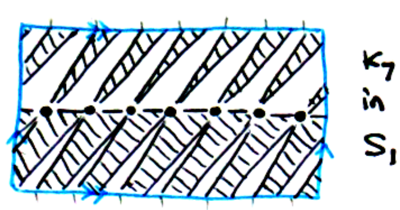
7. CONNECTIONS WITH DESIGN THEORY

Designs can often be used to help construct embeddings. Often need some kind of extra condition to make sure we get proper rotations.

Biembeddings of Steiner triple systems

If we have a 2-face-colourable triangular embedding of K_n, then each colour class forms a partition of the edges of K_n into triangles. In other words, we have a set of triples chosen from n elements so that every pair occurs in exactly one triple: a *Steiner triple system (STS)*. Altogether this is a *biembedding of Steiner triple systems*.

Example: 2-face-colourable embedding of K_7 on torus, shown below, is a biembedding of the Fano plane (the unique up to isomorphism STS of order 7) with itself.



In general if we just take two arbitrary Steiner triple systems then we do not get an embedding: we have a set of closed walks covering each edge twice, but may not have proper rotations.

If we take two Steiner triple systems T_1 and T_2, not clear when T_1 can be biembedded with something isomorphic to T_2. At least one case known where this cannot be done if we insist that the embedding must be orientable.

Biembeddings of Latin squares

Definition: A *latin square* is an $n \times n$ array of n symbols so that every symbol occurs exactly once in each row and each column.

Suppose we have a 2-face-colourable triangular embedding of a complete tripartite graph $K_{n,n,n}$ with tripartition (A, B, C) where $A = \{a_1, a_2, \ldots, a_n\}$, etc. Take one colour class of faces, then we have a partition of the edges of $K_{n,n,n}$ into triangles. If we interpret a triangle $(a_i b_j c_k)$ as telling us to put symbol k in row i, column j, then we get a latin square for each colour class. Altogether this is a *biembedding of latin squares*. If this exists, the surface is necessarily orientable.

Again, if we take two arbitrary latin squares, it is not clear if we can biembed them. But there is one positive result with very useful consequences.

Definition: A latin square L is *consecutive-row-hamiltonian* if for every two (cyclically) consecutive rows, the permutation we get by mapping symbols in the first row to the symbols in the same column in the second row is a cyclic (hamiltonian!) permutation.

Simple example: Z_n, the addition table of $\mathbb{Z}_n$, is consecutive-row-hamiltonian.

Theorem (Grannell and Griggs [GG08]): Any latin square that is consecutive-row-hamiltonian has a biembedding with something isomorphic to itself (in fact, to itself with all rows shifted up one position).

This was used as part of first construction of n^{an^2} nonisomorphic triangulations of K_n for certain n. Overall construction used ideas related to earlier result giving c^{n^2} such triangulations (mentioned in section on surgery).

Latin squares and hamilton cycle embeddings of complete tripartite graphs

Can also use latin squares to get other embeddings of complete tripartite graphs: ones where all facial walks are hamilton cycles. Need two conditions. First, latin square must be consecutive-entry-hamiltonian (similar to consecutive-row-hamiltonian, and in fact could use that instead). Second, latin square L must have an *orthogonal mate*: another latin square L' such that for every symbol s of L and every symbol s' of L' there is some row and column that contains s in L and s' in L'.

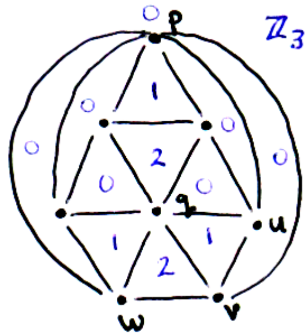
Theorem (Ellingham and Schroeder): An $n \times n$ latin square that is consecutive-entry-hamiltonian and has an orthogonal mate can be used to construct a 2-face-colourable hamilton cycle embedding of $K_{n,n,n}$. Every face has an *abc*-pattern (useful for tripling construction mentioned in section on surgery). If $n \geq 3$ is not twice a prime then such a latin square exists.

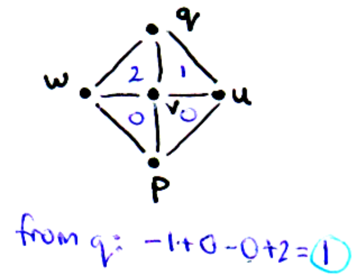
For n odd can again use Z_n, addition table of $\mathbb{Z}_n$. Much trickier for even n.

8. BOUCHET'S COVERING TRIANGULATIONS

Idea: Lift triangulation of G to triangulation of $G[\overline{K_m}] = G_{(m)}$, graph where we replace each $x \in V(G)$ by m independent vertices (x, i), $i \in \mathbb{Z}_m$, and $(x, i)(y, j) \in E(G_{(m)} \Leftrightarrow xy \in E(G)$, i.e. each edge is replaced by a copy of $K_{m,m}$. Original paper is [Bo78b].

Definition: Suppose G is eulerian (every vertex has even degree) and Ψ is a triangulation of G. Let $T = T(\Psi)$ be the set of triangles of Ψ. An *m-valuation* is a map $\phi : T \to \mathbb{Z}_m$. An m-valuation is *generative* if the alternating sum around every vertex is a generator of $\mathbb{Z}_m$.

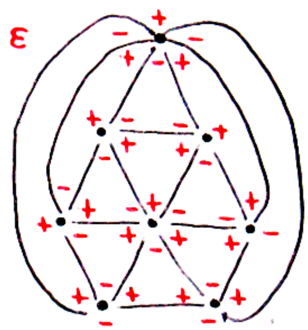


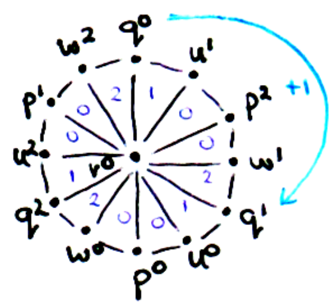


Formalization: Make a bit more precise: a corner of the embedding is represented by a (vertex, triangle) pair (x, t). Assign a sign $\varepsilon(x, t) \in \{-1, 1\}$ to each corner so that the signs alternate around every vertex. Define

$\overline{\phi}(x) = \sum_{x \in t} \varepsilon(x, t)\phi(t)$

for each vertex x. Then we want every $\overline{\phi}(x)$ to be a generator of $\mathbb{Z}_m$.





Theorem: If ϕ is a generative m-valuation then we have a triangulation of $G_{(m)}$ whose triangles are given by

$\{(\,(x, i)(y, j)(z, k)\,) \mid (xyz) \in T,\ i + j + k = \phi(t)\}$.

This has the same orientability as the original triangulation.

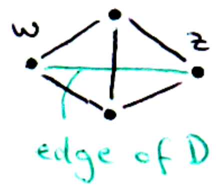
- Clear that we get two triangles containing every $(x, i)(y, j)$, corresponding to the two original triangles (wxy) and (xyz): values of i and j force values of h and k for third vertices (w, h) and (z, k).
- So just need to verify proper rotations. When we follow triangles around a vertex (x, i) from edge $(x, i)(y, j)$ will end up at edge $(x, i)(y, j \pm \overline{\phi}(x))$ after going around x once: since $\overline{\phi}(x)$ generates $\mathbb{Z}_m$, we end up with *all* neighbours of (x, i) after doing this m times.

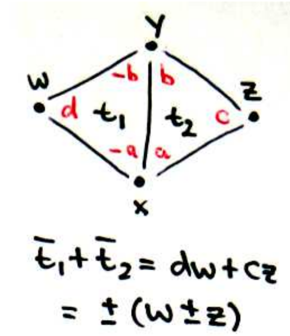
Finding a generative m-valuation

Restate question in more formal algebraic way.

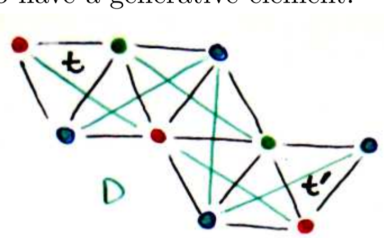
- Consider $\mathbb{Z}_m V$ = formal $\mathbb{Z}_m$-linear combinations of vertices in G, $\mathbb{Z}_m$-module.
- For each triangle t define $\overline{t} = \sum_{x \in t} \varepsilon(x, t)x \in \mathbb{Z}_m V$. Define $\overline{\phi} = \sum_{t \in T} \phi(t)\overline{t}$. ϕ is generative m-valuation if coefficient of $\overline{\phi}$ for vertex x is a generator of $\mathbb{Z}_m$ for all x: in that case say that $\overline{\phi}$ is *generative* element of $\mathbb{Z}_m V$. This coefficient is just what we called $\overline{\phi}(x)$ before: formal $\mathbb{Z}_m$-linear combinations are equivalent to $\mathbb{Z}_m$-valued functions.
- Define submodule $\overline{T}$ generated by $\{\overline{t} \mid t \in T\}$. Want to know if any generative element in $\overline{T}$.

- Depends on structure of *diagonal graph* $D = D(\Psi)$: $V(D) = V(G)$, join w and z if they are in adjacent triangles (wxy) and (xyz).





  - If $wz \in E(D)$ then one of $w+z$, $w-z$ is in $\overline{T}$: call it $\alpha(w,z)$.
  - If u and v are in the same component of D then one of $u+v$, $u-v$ is in $\overline{T}$: again call it $\alpha(u,v)$. (Use induction on previous statement.)
  - So if could partition each component of D into pairs of vertices (u_i, v_i), add up all $\alpha(u_i, v_i)$ and all coefficients ± 1, so have a generative element.

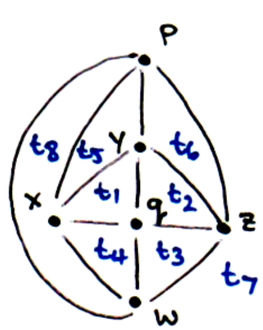


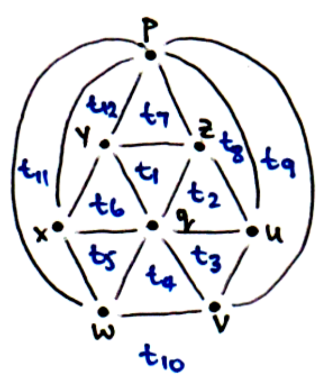
- What do components of D look like? For a given triangle t and $x \in T$, for any other triangle t' there is $x' \in t'$ such that x and x' are in the same component of D. So D has at most three components, and each component contains a fixed number of vertices of each triangle.

  For example, in the octahedron (as shown) there are three components of D with even vertex sets $\{p, q\}$, $\{w, y\}$, $\{x, z\}$. Take

$$\begin{aligned}(\overline{t_1} + \overline{t_5}) + (\overline{t_1} + \overline{t_4}) + (\overline{t_1} + \overline{t_2}) &= 3t_1 + t_2 + t_4 + t_5 \\ &= \pm(p \pm q) \pm (w \pm y) \pm (x \pm z)\end{aligned}$$

  which is generative for any m (even or odd).





Theorem: Suppose m is odd and Ψ is a triangulation of eulerian G. Then Ψ has a generative m-valuation and hence a triangular embedding of $G_{(m)}$ of the same orientability as Ψ.

Proof:

- Fix a triangle $t = (xyz)$. Choose ε values so $\varepsilon(x, t) = \varepsilon(y.t) = \varepsilon(z, t) = 1$.
- Partition each component of D into pairs of vertices, as follows:
  - if there is a leftover vertex make sure it is a vertex of t;
  - if a vertex of t is in one of the pairs (u_i, v_i), make sure it is u_i (so its coefficient is definitely 1, not -1).

    If all components of D are even, just add up all $\alpha(u_i, v_i)$ as mentioned above: all coefficients are ± 1.
- If some component of D is odd then adding up all $\alpha(u_i, v_i)$ will leave out some element(s) of t. So add up all $\alpha(u_i, v_i)$ and add $\overline{t} = x + y + z$. Now all coefficients ± 1 except possibly coefficients of 2 for x, y or z: since m is odd, still generative. ∎

Example: In $\overline{K_2} + C_6$ as shown, D has three components with vertex sets $\{p, q\}$, $\{u, w, y\}$ and $\{v, x, z\}$. Assuming all ε values of t_1 are $+1$, we take $\overline{t_1} + \alpha(q, p) + \alpha(u, w) + \alpha(v, x)$ which has coefficient 2 for q and coefficient ± 1 for everything else.

Note: As mentioned earlier, if all components of D are even order then works for any m; Bouchet gives other conditions that will guarantee this.

Folded coverings

If we want to extend Theorem above to even m, will be enough to do it for $m = 2$, then can use induction for powers of 2 and combine with result for odd m. But it can be shown that it is not always possible to get a generative 2-valuation.

Instead, need to use *folded coverings* [Bo82]. Original coverings have property that two triangles containing given edge $(x, i)(y, j)$ correspond to the two distinct triangles containing xy in G. But for folded covering, may have *fold* on edge $(x, i)(y, j)$: both triangles containing this edge correspond to same original triangle (xyz).

Theorem: Suppose Ψ is a triangulation of eulerian G. Then there is a triangular embedding of $G_{2)}$ of the same orientability as Ψ, obtained by a folded covering.

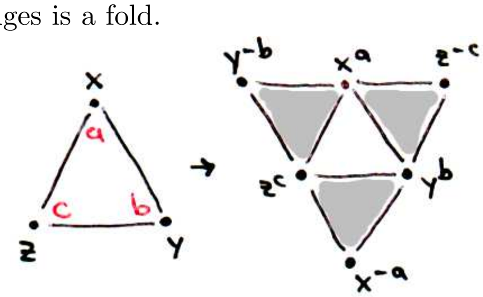
Proof: Assign $\varepsilon(x, t)$ values as previously (± 1 values at corners, alternating around each vertex).

- For each $x \in V(G)$ let $(x, -1)$ and $(x, 1)$ be corresponding vertices in $G_{(2)}$.
- Given a triangle $t = (xyz)$ in Ψ with $a = \varepsilon(x, t)$, $b = \varepsilon(y, t)$, $c = \varepsilon(z, t)$, replace by four triangles

  $(\,(x, a)(y, b)(z, c)\,)$ (primary triangle),

  $(\,(x, -a)(y, b)(z, c)\,)$, $(\,(x, a)(y, -b)(z, c)\,)$, $(\,(x, a)(y, b)(z, -c)\,)$ (three secondary triangles).

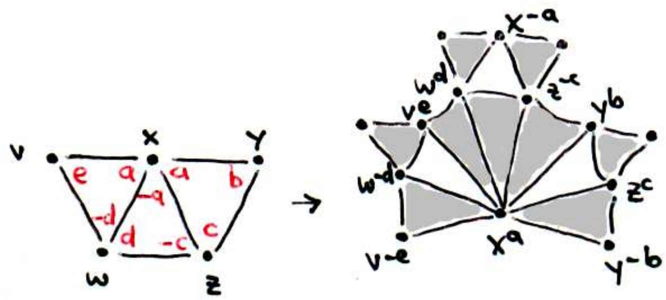
  Note that each edge $(x, a)(y, b)$, $(x, a)(z, c)$, $(y, b)(z, c)$ appears in two triangles coming from (xyz) so each of these edges is a fold.



- Each edge occurs in two triangles: suppose we also have original triangle $t' = (wxy)$. Then $(x, a)(y, b)$ occurs in two triangles from $t = (xyz)$; $(x, -a)(y, -b)$ occurs in two triangles from $t' = (wxy)$ (also a fold) because $\varepsilon(x, t') = -\varepsilon(x, t) = -a$ and $\varepsilon(y, t') = -\varepsilon(y, t) = -b$;

$(x, a)(y, -b)$ and $(x, -a)(y, b)$ each appear in one triangle from $t = (xyz)$ and one triangle from $t' = (wxy)$ (so not folds).

- Can follow triangles around each vertex $(x, \pm 1)$: close up because original degree of x was even, so have proper rotation.



- Map local orientation of triangles in Ψ to new triangulation: use same orientation for primary triangles, reverse for secondary triangles. Consistent if and only if original orientation consistent. ∎

Other important results by Bouchet and coauthors

Theorem: If Ψ is a triangulation of an eulerian complete multipartite graph then $G_{(m)}$ has a triangulation of the same orientability as Ψ, obtained using a generative m-valuation, for all $m \geq 2$. Proof shows that we can avoid odd order components of diagonal graph when m is even.

Theorem: If p is an odd prime and Ψ is a triangulation of a graph G such that Ψ^* (the dual of Ψ) has a nowhere zero p-flow then there is a triangular embedding of $G_{(p)}$ of the same orientability as Ψ.

Note: Since all 2-connected graphs have nowhere zero 6-flows (Seymour), can always do this for $p \geq 7$. If 5-flow conjecture is true, would always work for $p = 5$, too. In special cases can work for $p = 3$ or 5 (e.g., see below).

Theorem: If Ψ is a triangulation of a 4-colourable graph $G \not\cong K_4$, then we get a triangular embedding of $G_{(m)}$ of the same orientability as Ψ for $m = 3$ and hence (by repetition, and using the fact that a 4-face-colourable graph has a nowhere zero 4-flow) for all odd m.

Non-triangular embeddings

Bouchet's constructions are for triangulations. But can use, perhaps in modified form, if convert other embeddings into triangulations by adding extra edges or vertices. A couple of examples:

1. Lifting embeddings where all faces have even lengths, paper by Bouchet. First add a new vertex inside each face so we have an Eulerian triangulation. Now find an m-valuation ϕ so that values/coefficients of $\overline{\phi}$ are generators of $\mathbb{Z}_m$ for original vertices, but are 0 for new vertices.

2. In some cases it makes sense to just directly apply Bouchet's results after converting to a triangulation. For example, Ellingham and Schroeder [ES12] used Bouchet's results to help construct hamilton cycle embeddings of regular complete tripartite graphs:

   hamilton cycle embedding of $K_{t,t,t}$
   $\to$ triangulation of $K_{2t,t,t,t}$ (add vertex in each face)

   and apply Bouchet lifting to get

   triangulation of $K_{2mt,mt,mt,mt}$
   $\to$ hamilton cycle embedding of $K_{mt,mt,mt}$ (delete first vertex class).

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