Tight distance-regular graphs with classical parameters

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Distance-regular graphs

- Let Γ be a graph of diameter d with vertex set $V\Gamma$, and $\Gamma_i(u)$ be the set of vertices of Γ at distance i from $u \in V\Gamma$.

- For $u, v \in V\Gamma$ with $\partial(u, v) = h$, let the *intersection numbers* be

$$p_{ij}^h := |\Gamma_i(u) \cap \Gamma_j(v)| \ .$$

- The graph Γ is *distance-regular* if the values of $p_{ij}^h(u, v)$ only depend on the choice of distances h, i, j and not on the particular vertices u, v.

Intersection array

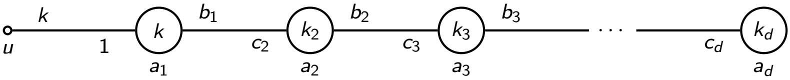
- Distance-regular graphs are regular with valency $k := p^0_{11}$ and have subconstituents $\Gamma_i(u)$ of size $k_i := p^0_{ii}$ and valency $a_i := p^i_{1,i}$ $(0 \le i \le d)$.

- All p^h_{ij} can be determined from the *intersection array*

$$\{k, b_1, \ldots, b_{d-1}; 1, c_2, \ldots, c_d\} ,$$

where $b_i := p^i_{1,i+1}$, $c_i := p^i_{1,i-1}$ and $a_i + b_i + c_i = k$ $(0 \le i \le d)$.

- Eigenvalues and their multiplicities can be computed directly from the intersection array.



Distance-regular graphs with classical parameters

- A. Neumaier [BCN89] observed that the intersection arrays of many known distance-regular graphs can be expressed in terms of just four parameters.

- A distance-regular graph of diameter d has *classical parameters* (d, b, α, β) if its intersection array satisfies

$$b_i = ([d] - [i])(\beta - \alpha[i]) \quad (0 \le i \le d-1) \quad \text{and}$$
$$c_i = [i](1 + \alpha[i-1]) \qquad (1 \le i \le d),$$

  where $[n] := [n]_b := \sum_{i=0}^{n-1} b^i$ is the b-analogue of n.

- The parameter b is an integer distinct from 0 and -1.

name	d	b	$\alpha+1$	$\beta+1$
Johnson graphs $J(e,d)$, $e \geq 2d$	d	1	2	$e-d+1$
Grassmann graphs $J_q(e,d)$, $e \geq 2d$	d	q	$q+1$	$[e-d+1]$
Twisted Grassmann graphs $\hat{J}_q(2d+1,d)$	d	q	$q+1$	$[d+2]$
Hamming graphs $H(d,e)$	d	1	1	e
Doob graphs $\hat{H}^i(d,4)$, $1 \leq i \leq d/2$	d	1	1	4
Halved cubes $\frac{1}{2}H(n,2)$	d	1	3	$m+1$
Dual polar graphs $B_d(q)$	d	q	1	$q+1$
Dual polar graphs $C_d(q)$	d	q	1	$q+1$
Dual polar graphs $D_d(q)$	d	q	1	2
Hemmeter graphs $\hat{D}_d(q)$	d	q	1	2
Halved dual polar graphs $D_{n,n}(q)$	d	q^2	$[3]_q$	$[m+1]_q$
Ustimenko graphs $\hat{D}_{n,n}(q)$	d	q^2	$[3]_q$	$[m+1]_q$
Dual polar graphs ${}^2D_{d+1}(q)$	d	q	1	q^2+1
Dual polar graphs ${}^2A_{2d}(q)$	d	q^2	1	q^3+1
Dual polar graphs ${}^2A_{2d-1}(q)$	d	q^2	1	$q+1$
or	d	$-q$	$\frac{1+q^2}{1-q}$	$\frac{1-(-q)^{d+1}}{1-q}$
Bilinear forms graphs $H_q(d,e)$, $e \geq d$	d	q	q	q^e
Alternating forms graphs $\mathrm{Alt}_n(q)$	d	q^2	q^2	q^m
Quadratic forms graphs $Q_{n-1}(q)$	d	q^2	q^2	q^m
Hermitean forms graphs $\mathrm{Her}_d(q)$	d	$-q$	$-q$	$-(-q)^d$
Triality graphs ${}^3D_{4,2}(q)$	3	$-q$	$\frac{1}{1-q}$	$[3]_q$
Affine $E_6(q)$ graphs	3	q^4	q^4	q^9
Exceptional Lie graphs $E_{7,7}(q)$	3	q^4	$[5]_q$	$[10]_q$
Gosset graph $E_7(1)$	3	1	5	10
Witt graph M_{23}	3	-2	-1	6
Witt graph M_{24}	3	-2	-3	11
Coset graph of the extended ternary Golay code	3	-2	-2	9

q is a prime power; $m = n = 2d+1$ or $m+1 = n = 2d$

Open cases

- For many known graphs with classical parameters, uniqueness is not known.
- There are also many open cases.
- All known open cases with diameter at least 4 have either $\alpha = b - 1$ or $\alpha = b$ [BCN89, Bro11].
- We have proven nonexistence for the cases
  - $(d, b, \alpha, \beta) = (3, 2, 1, 5)$ with 216 vertices [JV12],
  - $(d, b, \alpha, \beta) = (3, 3, 2, 10)$ with 1331 vertices, and
  - $(d, b, \alpha, \beta) = (3, 8, 7, 66)$ with 300763 vertices.

Tight distance-regular graphs

A. Jurišić, J. H. Koolen and P. Terwilliger [JKT00] established the *fundamental bound* for distance-regular graphs:

$$\left(\theta_1 + \frac{k}{a_1+1}\right)\left(\theta_d + \frac{k}{a_1+1}\right) \geq -\frac{ka_1 b_1}{(a_1+1)^2}.$$

A non-bipartite graph with equality in this bound is called *tight*. Such graphs can be parametrized with $d+1$ parameters.

The only known primitive tight graph is the Patterson graph with 22880 vertices, which is uniquely determined by its intersection array $\{280, 243, 144, 10; 1, 8, 90, 280\}$ [BJK08].

Local graphs of tight graphs

Theorem [JKT00, BCN89]:
For any vertex u of a tight distance-regular graph Γ,
the local graph $\Gamma(u)$ is strongly regular with nontrivial eigenvalues

$$\tau = -1 - \frac{b_1}{1+\theta_d} \quad \text{and} \quad \sigma = -1 - \frac{b_1}{1+\theta_1}$$

and multiplicities

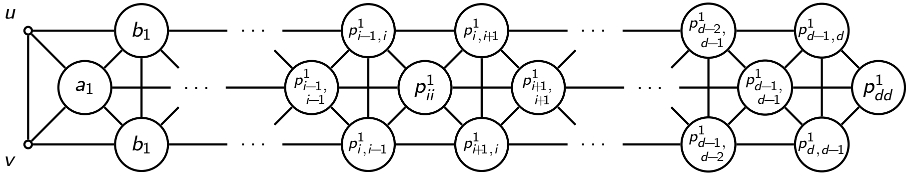
$$m_\tau = \frac{a_1(a_1-\sigma)(\sigma+1)}{(a_1+\sigma\tau)(\sigma-\tau)} \quad \text{and} \quad m_\sigma = \frac{a_1(a_1-\tau)(\tau+1)}{(a_1+\sigma\tau)(\tau-\sigma)}.$$

1-homogeneity

A partition $\{C_i\}_{i=1}^t$ of $V\Gamma$ is *equitable* if there exist parameters n_{ij} such that every vertex in C_i has precisely n_{ij} neighbours in C_j.

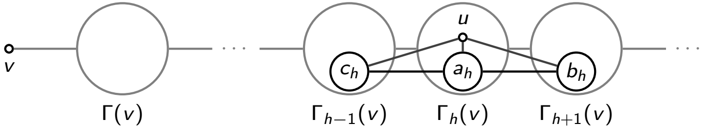
A graph is distance-regular *iff* the distance partition for every vertex is equitable with the same parameters.

A graph Γ is *1-homogeneous* [Nom94] if any partition of the graph corresponding to the distances from two adjacent vertices is equitable with the same parameters.



The CAB property

A graph Γ has the *CAB property* [JK00] if for any two vertices $u, v \in V\Gamma$, the partition of the local graph $\Gamma(u)$ corresponding to the distances from v is equitable with parameters only depending on the distance $\partial(u, v)$.



Characterization

Theorem [JKT00, JK00]: Let Γ be a distance-regular graph with $a_1 \neq 0$ and $a_d = 0$. The following are equivalent:

- Γ is 1-homogeneous,
- Γ has the CAB property,
- Γ is tight.

Tight distance-regular graphs with classical parameters

Proposition: A distance regular graph with classical parameters (d, b, α, β) and $d \geq 3$ is tight iff

$$\beta = 1 + \alpha[d-1] \quad \text{and} \quad b, \alpha > 0.$$

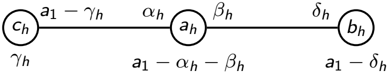
All known examples have $b = 1$:

- halved cubes $\frac{1}{2}H(2d, 2)$, $(d, b, \alpha, \beta) = (d, 1, 1, d)$,
- Johnson graphs $J(2d, d)$, $(d, b, \alpha, \beta) = (d, 1, 2, 2d-1)$, and
- the Gosset graph $E_7(1)$, $(d, b, \alpha, \beta) = (3, 1, 4, 9)$.

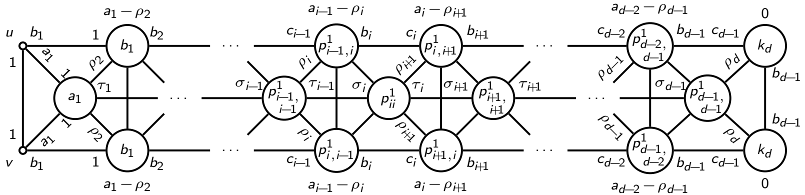
These graphs are uniquely determined by their parameters.

Parameters of partitions

The parameters of the CAB and 1-homogeneous partitions of a tight distance-regular graphs with classical parameters can be computed explicitly:



$\alpha_h = 1+\alpha[h-1], \quad \beta_h = b(1+\alpha b^h[d-h-1]), \quad \gamma_h = \delta_{h-1} = \alpha(b+1)[h-1],$



$\rho_i = \alpha b^{i-2}(b+1)[i-1], \quad \sigma_i = [i-1](1+\alpha[i-1]), \quad \tau_i = b^{i+1}[d-i-1](1+\alpha b^i[d-i-1]).$

A feasible family

We find a two-parametrical family of classical parameters for tight distance-regular graphs:

$$(d, b, \alpha, \beta) = (d, b, b-1, b^{d-1}). \quad (1)$$

- $\alpha = b - 1$ implies that corresponding graphs are formally self-dual.
- For $b = 1$ we get d-cubes, which are bipartite and thus not tight.
- For $d = 3$, the parameters are not feasible as they imply $p^3_{33} < 0$.
- For $b \geq 2$ and $d \geq 4$ we have a feasible parameter set for a primitive distance-regular graph.

Theorem: A graph with classical parameters (1) and $b \geq 2$, $d \geq 4$ does not exist.

Idea of proof: local graphs are strongly regular, but their eigenvalues have nonintegral multiplicities.

Local graphs

Let Γ be a tight distance-regular graph of diameter $d \geq 4$ with classical parameters (d, b, α, β), where $b \geq 2$ and $\alpha \in \{b, b+1\}$.

- The local graphs of Γ have parameters of Latin square and Steiner system graphs, respectively.
- We have checked that the multiplicity of the smallest eigenvalue is never integral when $\alpha = b$ and $d \leq 17$, or $\alpha = b + 1$ and $d \leq 5$.

Conjecture: The local graph of a tight distance-regular graph with classical parameters (d, b, α, β), where $d \geq 3$ and $b \geq 2$, is not a Latin square or Steiner system graph.

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