Generalized Cayley graphs

Ademir Hujdurović (University of Primorska)

Joint work with Klavdija Kutnar, Paweł Petecki and Anastasiya Tanana.

International Conference on Graph Theory

27.5.2015

Overview

- Cayley graphs
- Generalized Cayley graphs
- Automorphism of Generalized Cayley graphs
- Non-Cayley vertex-transitive generalized Cayley graphs

Basics about graphs

A graph is an ordered pair $\Gamma = (V, E)$, where V denotes the set of vertices, and E denotes the set of edges of the graph Γ.

Automorphism of a graph Γ is a bijective function $\varphi : V(\Gamma) \to V(\Gamma)$ such that $\{x, y\} \in E(\Gamma) \Leftrightarrow \{\varphi(x), \varphi(y)\} \in E(\Gamma)$.

We define the set $Aut(\Gamma)$ to be the set of all automorphisms of the graph Γ.
It is not difficult to see that $Aut(\Gamma)$ is in fact the group with respect to composition of functions, and it is called the automorphism group of Γ.

Actions on graphs

When we speak about graphs we consider the action of the group of the automorphisms on the graph.

Actions on graphs

When we speak about graphs we consider the action of the group of the automorphisms on the graph. We say that a graph Γ is

- vertex-transitive $\Leftrightarrow$ $\mathrm{Aut}(\Gamma)$ acts transitively on the vertex set of the graph;
- edge-transitive $\Leftrightarrow$ $\mathrm{Aut}(\Gamma)$ acts transitively on the edge set of the graph;
- arc-transitive $\Leftrightarrow$ $\mathrm{Aut}(\Gamma)$ acts transitively on the arc set of the graph.

Given a group G and a subset S of G such that:

(i) $1 \notin S$,

(ii) $S^{-1} = S$;

the *Cayley graph* $\mathrm{Cay}(G, S)$ of G relative to S has vertex set G and edges of the form $\{g, gs\}$ where $g \in G$ and $s \in S$.

Given a group G and a subset S of G such that:

(i) $1 \notin S$,

(ii) $S^{-1} = S$;

the *Cayley graph* $\mathrm{Cay}(G, S)$ of G relative to S has vertex set G and edges of the form $\{g, gs\}$ where $g \in G$ and $s \in S$.

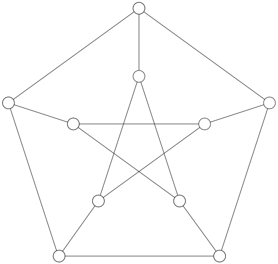
Example

$G = \mathbb{Z}_{10}$, $S = \{\pm 1, 5\}$.

Cayley graphs

If $X = Cay(G, S)$ then the action of G on itself by the left multiplication induces a subgroup of the automorphism group which acts transitively on vertices, hence every Cayley graph is vertex-transitive.

If $X = Cay(G, S)$ then the action of G on itself by the left multiplication induces a subgroup of the automorphism group which acts transitively on vertices, hence every Cayley graph is vertex-transitive.
However, not every vertex-transitive graph is Cayley graph. The smallest vertex-transitive graph which is not Cayley is the Petersen graph.



Let G be a finite group, S a non-empty subset of G and α an automorphism of G such that the following conditions are satisfied:

(i) $\alpha^2 = 1$,

(ii) $\alpha(g^{-1})g \notin S$, $(\forall g \in G)$

(iii) $\alpha(S^{-1}) = S$.

Then the *generalized Cayley graph* $X = GC(G, S, \alpha)$ on G with respect to the ordered pair (S, α) is a graph with vertex set G, and edges of form $\{g, \alpha(g)s\}$, where $g \in G$ and $s \in S$.

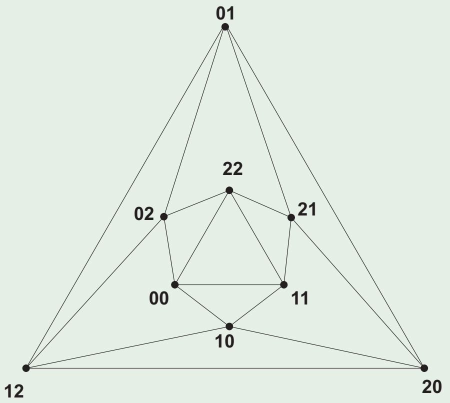
Example

$G = \mathbb{Z}_{10}$, $\alpha(x) = -x$, $S = \{\pm 1, 5\}$.

Example

Example

Let $G = \mathbb{Z}_3 \times \mathbb{Z}_3$, $S = \{(1,0),(1,1),(0,2),(2,2)\}$ and $\alpha : (i,j) \mapsto (j,i)$.



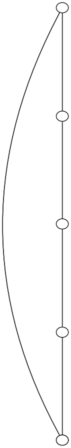
History of generalized Cayley graphs

The concept of generalized Cayley graphs was introduced by Marušič, Scapellato and Zagaglia Salvi in 1992. They studied properties of such graphs relative to double covers of graphs (sometimes called bipartite double cover or canonical double cover).

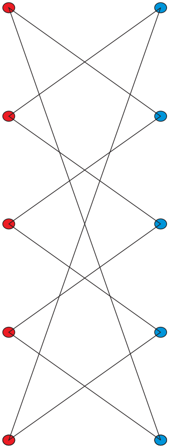
History of generalized Cayley graphs

The concept of generalized Cayley graphs was introduced by Marušič, Scapellato and Zagaglia Salvi in 1992. They studied properties of such graphs relative to double covers of graphs (sometimes called bipartite double cover or canonical double cover).

Double cover $B(X)$ of a graph X is the direct product $X \times K_2$. This means that $V(B(X)) = V(X) \times \mathbb{Z}_2$ and all the edges of $B(X)$ are $\{(x,0),(y,1)\}$ and $\{(x,1),(y,0)\}$ where $\{x,y\}$ is an edge in X.



C_5



$B(C_5)$

It is easily seen that $\mathrm{Aut}(B(X))$ contains a subgroup isomorphic to $\mathrm{Aut}(X) \times \mathbb{Z}_2$. If $\mathrm{Aut}(B(X))$ is isomorphic to $\mathrm{Aut}(X) \times \mathbb{Z}_2$ then the graph X is called **stable**, otherwise it is called **unstable**.

It is easily seen that $\mathrm{Aut}(B(X))$ contains a subgroup isomorphic to $\mathrm{Aut}(X) \times \mathbb{Z}_2$. If $\mathrm{Aut}(B(X))$ is isomorphic to $\mathrm{Aut}(X) \times \mathbb{Z}_2$ then the graph X is called **stable**, otherwise it is called **unstable**.

Theorem (Marušič, Scapellato, Zagaglia Salvi, 1992)

Let X be a non-bipartite graph. Then its double cover is a Cayley graph if and only if X is a generalized Cayley graph.

It is easily seen that $\mathrm{Aut}(B(X))$ contains a subgroup isomorphic to $\mathrm{Aut}(X) \times \mathbb{Z}_2$. If $\mathrm{Aut}(B(X))$ is isomorphic to $\mathrm{Aut}(X) \times \mathbb{Z}_2$ then the graph X is called **stable**, otherwise it is called **unstable**.

Theorem (Marušič, Scapellato, Zagaglia Salvi, 1992)

Let X be a non-bipartite graph. Then its double cover is a Cayley graph if and only if X is a generalized Cayley graph.

Proposition (Marušič, Scapellato, Zagaglia Salvi, 1992)

Let X be a generalized Cayley graph. If X is stable, then it is a Cayley graph.

Therefore, every generalized Cayley graph which is not Cayley graph is unstable.

Lemma (H, Kutnar, Marušič, 2015)

Let $X = GC(G, S, \alpha)$ be a generalized Cayley graph on a group G with respect to the ordered pair (S, α), and let $Fix(\alpha) = \{g \in G \mid \alpha(g) = g\}$. Then $Fix(\alpha)_L \leq \mathrm{Aut}(X)$ and moreover it acts semiregularly on $V(X)$.

Lemma (H, Kutnar, Marušič, 2015)

Let $X = GC(G, S, \alpha)$ be a generalized Cayley graph on a group G with respect to the ordered pair (S, α), and let $Fix(\alpha) = \{g \in G \mid \alpha(g) = g\}$. Then $Fix(\alpha)_L \leq \mathrm{Aut}(X)$ and moreover it acts semiregularly on $V(X)$.

Theorem (H, Kutnar, Marušič, 2015)

Let $X = GC(G, S, \alpha)$ be a generalized Cayley graph on a group G with respect to the ordered pair (S, α). Then there exists a non-trivial element $g \in G$, which is fixed by α. Moreover, X admits a semiregular automorphism which lies in $G_L \cap Aut(X)$.

Automorphisms of a Generalized Cayley graphs

Let $X = Cay(G, S)$ be a Cayley graph, and let $\mathrm{Aut}(G, S)$ denote the set of all automorphisms of G that fix set S, that is

$$\mathrm{Aut}(G, S) = \{\varphi \in \mathrm{Aut}(G) \mid \varphi(S) = S\}.$$

Automorphisms of a Generalized Cayley graphs

Let $X = Cay(G, S)$ be a Cayley graph, and let $\mathrm{Aut}(G, S)$ denote the set of all automorphisms of G that fix set S, that is

$$\mathrm{Aut}(G, S) = \{\varphi \in \mathrm{Aut}(G) \mid \varphi(S) = S\}.$$

It is well-known that $\mathrm{Aut}(G, S)$ is the subgroup of $\mathrm{Aut}(X)$. Moreover, it is contained in the stabilizer of $1_G \in V(X)$.

Let $X = Cay(G, S)$ be a Cayley graph, and let $\mathrm{Aut}(G, S)$ denote the set of all automorphisms of G that fix set S, that is

$$\mathrm{Aut}(G, S) = \{\varphi \in \mathrm{Aut}(G) \mid \varphi(S) = S\}.$$

It is well-known that $\mathrm{Aut}(G, S)$ is the subgroup of $\mathrm{Aut}(X)$. Moreover, it is contained in the stabilizer of $1_G \in V(X)$. Motivated by the above result, let us define

$$\mathrm{Aut}(G, S, \alpha) = \{\varphi \in \mathrm{Aut}(G) \mid \varphi(S) = S,\ \alpha\varphi = \varphi\alpha\}.$$

Let $X = Cay(G, S)$ be a Cayley graph, and let $\mathrm{Aut}(G, S)$ denote the set of all automorphisms of G that fix set S, that is

$$\mathrm{Aut}(G, S) = \{\varphi \in \mathrm{Aut}(G) \mid \varphi(S) = S\}.$$

It is well-known that $\mathrm{Aut}(G, S)$ is the subgroup of $\mathrm{Aut}(X)$. Moreover, it is contained in the stabilizer of $1_G \in V(X)$. Motivated by the above result, let us define

$$\mathrm{Aut}(G, S, \alpha) = \{\varphi \in \mathrm{Aut}(G) \mid \varphi(S) = S,\ \alpha\varphi = \varphi\alpha\}.$$

Theorem (H, Kutnar, Marušič, 2015)

Let $X = GC(G, S, \alpha)$ be a generalized Cayley graph on a group G with respect to the ordered pair (S, α). Then $\mathrm{Aut}(G, S, \alpha) \leq \mathrm{Aut}(X)$ which fixes the vertex $1_G \in V(X)$.

Vertex-transitive generalized Cayley non Cayley graphs

Theorem (H, Kutnar, Marušič, 2015)

For a natural number $k \geq 1$ *let* $n = 2((2k+1)^2+1)$ *and let* X *be the generalized Cayley graph* $GC(\mathbb{Z}_n, S, \alpha)$ *on the cyclic group* $\mathbb{Z}_n$ *with respect to* $S = \{\pm 2, \pm 4k^2, 2k^2+2k+1\}$ *and the automorphism* $\alpha \in \mathrm{Aut}(\mathbb{Z}_n)$ *defined by the rule* $\alpha(x) = ((2k+1)^2+2) \cdot x$. *Then* X *is a non-Cayley vertex-transitive graph.*

Vertex-transitive generalized Cayley non Cayley graphs

Theorem (H, Kutnar, Marušič, 2015)

For a natural number $k \geq 1$ let $n = 2((2k+1)^2 + 1)$ and let X be the generalized Cayley graph $GC(\mathbb{Z}_n, S, \alpha)$ on the cyclic group $\mathbb{Z}_n$ with respect to $S = \{\pm 2, \pm 4k^2, 2k^2 + 2k + 1\}$ and the automorphism $\alpha \in \mathrm{Aut}(\mathbb{Z}_n)$ defined by the rule $\alpha(x) = ((2k+1)^2 + 2) \cdot x$. Then X is a non-Cayley vertex-transitive graph.

Theorem (H, Kutnar, Marušič, 2015)

For a natural number k such that $k \not\equiv 2 \pmod 5$, $t = 2k + 1$ and $n = 20t$, the generalized Cayley graph $GC(\mathbb{Z}_n, S, \alpha)$ on the cyclic group $\mathbb{Z}_n$ with respect to $S = \{\pm 2t, \pm 4t, 5, 10t - 5\}$ and the automorphism $\alpha \in \mathrm{Aut}(\mathbb{Z}_n)$ defined by the rule $\alpha(x) = (10t + 1)x$, is a non-Cayley vertex-transitive graph.

Let $\omega_\alpha \colon G \to G$ be the mapping defined by $\omega_\alpha(x) = \alpha(x)x^{-1}$ and let $\omega_\alpha(G) = \{\omega_\alpha(g) \mid g \in G\}$. Notice that the definition of generalized Cayley graphs (ii) is equivalent to $\omega_\alpha(G) \cap S = \emptyset$.

Group automorphisms of order 2

Let $\omega_\alpha \colon G \to G$ be the mapping defined by $\omega_\alpha(x) = \alpha(x)x^{-1}$ and let $\omega_\alpha(G) = \{\omega_\alpha(g) \mid g \in G\}$. Notice that the definition of generalized Cayley graphs (ii) is equivalent to $\omega_\alpha(G) \cap S = \emptyset$.

Proposition

(a) *If G is an Abelian group then $\omega_\alpha(G)$ is a subgroup of G;*

(b) *for every $x \in \omega_\alpha(G)$, it holds $\alpha(x) = x^{-1}$.*

Theorem (Miller, 1909)

If G is an Abelian group of odd order and $\alpha \in \mathrm{Aut}(G)$ such that $\alpha^2 = 1$ then $G = \mathrm{Fix}(\alpha) \times \omega_\alpha(G)$.

The previous result enables us to describe generalized Cayley graphs on an Abelian group of odd order in a simple way. If G is an Abelian group of odd order, and $\alpha \in \mathrm{Aut}(G)$ such that $\alpha^2 = 1$, then we can write $G = G_1 \times G_2$, where $G_1 = \mathrm{Fix}(\alpha)$, $G_2 = \omega_\alpha(G)$. Then for $(x, y) \in G_1 \times G_2$ we have $\alpha(x_1, x_2) = (x_1, x_2^{-1})$. Then $GC(G, S, \alpha)$ is isomorphic to the graph with vertex set $G_1 \times G_2$. Let $S' \subseteq G_1 \times G_2$ be the image of S. Then it is easy to see that:

(i) $S' \cap (\{1_{G_1}\} \times G_2) = \emptyset$;

(ii) $(s_1, s_2) \in S' \Leftrightarrow (s_1^{-1}, s_2) \in S'$.

The edges are now given with $(x_1, x_2) \sim (x_1 s_1, x_2^{-1} s_2)$, where $(s_1, s_2) \in S'$.

Theorem

Let $G = \mathbb{Z}_{2^n} \times H$, where H is an Abelian group of odd order and let $\alpha \in \mathrm{Aut}(G)$ such that $\alpha^2 = 1$. Then $H = H_1 \times H_2$, $\alpha(x, y_1, y_2) = (ax, y_1, y_2^{-1})$, where $a \in \{\pm 1, 2^{n-1} \pm 1\}$.

$\alpha(x) = -x$

$\alpha(x) = -x$

Theorem

Let $m \in \mathbb{N}$ and let H be a finite Abelian group of odd order. Then any generalized Cayley graph $X = GC(\mathbb{Z}_{2^m} \times H, S, \alpha)$ is a Cayley graph on $Dih(\mathbb{Z}_{2^{m-1}} \times H)$.

$\alpha(x) = -x$

Theorem

Let $m \in \mathbb{N}$ and let H be a finite Abelian group of odd order. Then any generalized Cayley graph $X = GC(\mathbb{Z}_{2^m} \times H, S, \alpha)$ is a Cayley graph on $Dih(\mathbb{Z}_{2^{m-1}} \times H)$.

Theorem

Let G be an Abelian group and α inversion automorphism of G. Then every generalized Cayley graph on G with respect to α is Cayley if and only if one of the following holds:

(i) *G is elementary Abelian 2-group;*

(ii) *Sylow 2-subgroup of G is cyclic.*

Order $2p$

Theorem

Every generalized Cayley graph of order $2p$ is a Cayley graph.

Theorem

Every generalized Cayley graph of order $2p$ is a Cayley graph.

Proof.

There are only two groups of order $2p$, $\mathbb{Z}_{2p}$ and D_{2p}.

Order $2p$

Theorem

Every generalized Cayley graph of order 2p is a Cayley graph.

Proof.

There are only two groups of order $2p$, $\mathbb{Z}_{2p}$ and D_{2p}.
If $G = \mathbb{Z}_{2p}$, then $\alpha = id$ or $\alpha(x) = -x$.

Order $2p$

Theorem

Every generalized Cayley graph of order $2p$ is a Cayley graph.

Proof.

There are only two groups of order $2p$, $\mathbb{Z}_{2p}$ and D_{2p}.
If $G = \mathbb{Z}_{2p}$, then $\alpha = id$ or $\alpha(x) = -x$.
Let $G = D_{2p} = \langle \tau, \rho \mid \tau^2 = \rho^p = id,\ \tau\rho\tau = \rho^{-1} \rangle$.

$$\alpha(\rho) = \rho^k \text{ where } (k, p) = 1;$$
$$\alpha(\tau) = \tau\rho^l.$$

Theorem

Every generalized Cayley graph of order $2p$ is a Cayley graph.

Proof.

There are only two groups of order $2p$, $\mathbb{Z}_{2p}$ and D_{2p}.
If $G = \mathbb{Z}_{2p}$, then $\alpha = id$ or $\alpha(x) = -x$.
Let $G = D_{2p} = \langle \tau, \rho \mid \tau^2 = \rho^p = id,\ \tau\rho\tau = \rho^{-1} \rangle$.

$$\alpha(\rho) = \rho^k \text{ where } (k, p) = 1;$$
$$\alpha(\tau) = \tau\rho^l.$$

$\alpha^2 = 1$ implies that $k \equiv \pm 1 \pmod{p}$ and $l(k+1) \equiv 0 \pmod{p}$. □

Thank you!!!