Spectral characterization of Signed Graphs

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Signed Graphs

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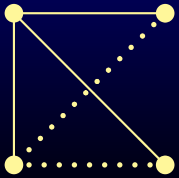


Positive edges are bold lines, negative edges are dotted lines.

Signature Switching

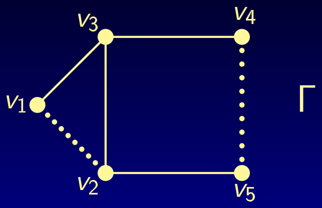
Definition

Let $\Gamma = (G, \sigma)$ be a signed graph and $U \subseteq V(G)$. The signed graph Γ^U obtained by negating the edges in the cut $[U; U^c]$ is a (sign) switching *of Γ. We also say that the signatures of Γ^U and Γ are* equivalent*.*

The signature switching preserves the set of the positive cycles.

We say that two signed graphs are *switching isomorphic* if their underlying graphs are isomorphic and the signatures are switching equivalent. The set of signed graphs switching isomorphic to Γ is the *switching isomorphism class* of Γ, written $[\Gamma]$.

Example of switching equivalent graphs



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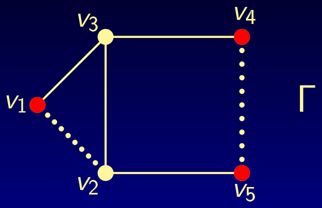

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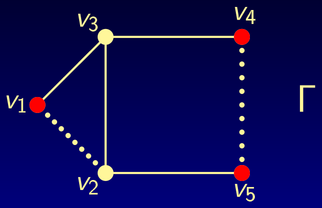

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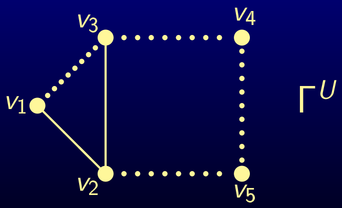


Note that the switching preserves the sign of the cycles!

Matrices of signed graphs

Let $M = M(G)$ be a *graph matrix* defined in a prescribed way. The *M-polynomial* of G is defined as $\det(\lambda I - M)$, where I is the identity matrix. The *M-spectrum* of G is a multiset consisting of the eigenvalues of $M(G)$. The largest eigenvalue of $M(G)$ is called the *M-spectral radius* of G.

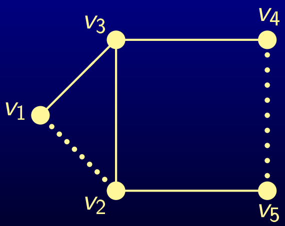
The graph matrices considered for an unsigned graph G are: the adjacency matrix $A(G)$, the Laplacian matrix $L(G) = D(G) - A(G)$, the signless Laplacian matrix $Q(G) = D(G) + A(G)$, where $D(G) = \mathrm{diag}(d_1, d_2, \ldots, d_n)$, and their normalized variants.

The adjacency matrix and the Laplacian matrix can be defined also for signed graphs.

Adjacency matrix of Signed Graphs

The adjacency matrix is defined as $A(\Gamma) = (a_{ij})$, where

$$a_{ij} = \begin{cases} \sigma(v_i v_j), & \text{if } v_i \sim v_j; \\ 0, & \text{if } v_i \nsim v_j. \end{cases}$$

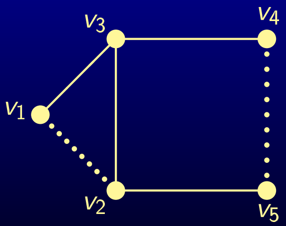


$$\rightsquigarrow \quad A(\Gamma) = \begin{pmatrix} 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 \end{pmatrix}$$

Laplacian of Signed Graphs

The Laplacian matrix of $\Gamma = (G, \sigma)$ is defined as $L(\Gamma) = D(G) - A(\Gamma) = (l_{ij})$

$$l_{ij} = \begin{cases} \deg(v_i), & \text{if } i = j; \\ -\sigma(v_i v_j), & \text{if } i \neq j. \end{cases}$$



$$\rightsquigarrow \quad L(\Gamma) = \begin{pmatrix} 2 & 1 & -1 & 0 & 0 \\ 1 & 3 & -1 & 0 & -1 \\ -1 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 2 & 1 \\ 0 & -1 & 0 & 1 & 2 \end{pmatrix}$$

Switching and signature similarity

Switching has a matrix counterpart. In fact, Let Γ and $\Gamma' = \Gamma^U$ be two switching equivalent graphs.
Consider the matrix $S_U = \mathrm{diag}(s_1, s_2, \dots, s_n)$ such that

$$s_i = \begin{cases} +1, & i \in U \\ -1, & i \in \Gamma \setminus U \end{cases}$$

S_U is called a *signature matrix* (or *state* matrix).

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It is easy to check that

$$A(\Gamma^U) = S_U\ A(\Gamma)\ S_U \quad \text{and} \quad L(\Gamma^U) = S_U\ L(\Gamma)\ S_U.$$

Hence, signed graphs from the same switching class share similar graph matrices, or switching isomorphic graphs are cospectral.

Coefficient Theorem for the adjacency polynomial

In the 1970's many researcher gave a combinatorial expression for the coefficients of the adjacency polynomial of a multi-di-graph. Here is the variant for signed graphs.

The elementary figures are the graphs K_2 and C_n; a basic figure is the disjoint union of elementary figures. Let $\mathcal{B}_i$ be the set of basic figures on i vertices, $p(B)$ # components of B, $|c(B)|$ # of cycles in B, and $\sigma(B) = \prod_{C \in c(B)} \sigma(C)$.

Theorem

Let Γ be a signed graph and let $\phi(\Gamma, x) = \sum_{i=0}^{n} a_i x^{n-1}$ be its adjacency characteristic polynomial. Then

$$a_i = \sum_{B \in \mathcal{B}_i} (-1)^{p(B)} 2^{|c(B)|} \sigma(B),$$

Coefficient Theorem for the Laplacian polynomial

A signed TU-graph is a graph whose components are trees or unbalanced unicyclic graphs (the unique cycle has a negative sign). If $H = T_1 \cup \cdots \cup T_r \cup U_1 \cup \cdots \cup U_s$, then $\gamma(H) = 4^s \prod_{i=1}^{r} |T_i|$.

Theorem

Let Γ be a signed graph and $\psi(L(\Gamma), x) = \sum_{i=0}^{n} b_i x^{n-i}$ be its Laplacian polynomial. Then we have:

$$b_i = (-1)^i \sum_{H \in \mathcal{H}_i} \gamma(H),$$

where $\mathcal{H}_i$ is the set number of signed TU-graphs on i edges.

Note: the signatures of bridges have no influence on the characteristic polynomial.

Proof

Observe that

$$\begin{aligned} BB^\top &= L(\Gamma), \\ B^\top B &= 2I + A_{\mathcal{L}(\Gamma)}, \end{aligned}$$

where $A_{\mathcal{L}(\Gamma)} = A(\mathcal{L}(\Gamma))$ is the adjacency matrix of $\mathcal{L}(\Gamma)$, the signed Line Graph of Γ.

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From the MacLaurin development we have that

$$\begin{aligned} \psi(\Gamma, x) &= x^{n-m}\phi(A_{\mathcal{L}(\Gamma)}, x-2) \\ &= x^{n-m}\sum_{k=0}^{m}\phi^{(k)}(A_{\mathcal{L}(\Gamma)}, -2)\frac{x^k}{k!} \\ &= x^{n-m}\sum_{k=m-n}^{m} x^k \frac{1}{k!}\phi^{(k)}(A_{\mathcal{L}(\Gamma)}, -2), \end{aligned}$$

Proof (continuation)

$$\psi(\Gamma, x) = x^{m-n} \sum_{k=m-n}^{m} x^k \sum_{|S|=k} \phi(A_{\mathcal{L}(\Gamma)-S}, -2).$$

Signed line graphs have -2 as an eigenvalue unless all components are line graphs of trees or unbalanced unicyclic graphs.

$$\sum_{|S|=k} \phi(A_{\mathcal{L}(\Gamma)-S}, -2) = (-1)^{m-k} \sum_{H \in \mathcal{H}_{m-k}} w(H).$$

$$\psi(\Gamma, x) = x^{m-n} \sum_{k=m-n}^{m} x^k (-1)^{m-k} \sum_{H \in \mathcal{H}_{m-k}} w(H),$$

and by putting $i = m - k$ we have

$$\psi(\Gamma, x) = \sum_{i=0}^{n} x^{n-i} (-1)^i \sum_{H \in \mathcal{H}_i} w(H).$$

Spectral determination problems

One of the oldest problems in Spectral Graph Theory is to establish whether a given graph G admits cospectral non-isomorphic graphs, w.r.t. some prescribed graph matrix M. In fact, we say that a graph G is M-DS iff any cospectral graph is isomorphic as well.

Spectral determination problems

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An analogous definition can be considered in the setting of signed graphs:

Definition

A signed graph Γ is said to be determined by the spectrum of the matrix $M(G)$ (in short, M-DS) iff any cospectral signed graphs is switching isomorphic as well.

In the sequel the L-cospectral relation will be denoted by $\sim$.

Laplacian Spectral Moments

Let $T_k = \sum_{i=1}^{k} \mu_i^k$ $(k = 0, 1, 2, \ldots)$ be the k-th spectral moment for the Laplacian spectrum of a signed graph Γ.

Theorem

Let $\Gamma = (G, \sigma)$ be a signed graph with n vertices, m edges, t^+ balanced triangles, t^- unbalanced triangles, and vertex degrees $d_1, d_2, \ldots, d_n$. We have

$$T_1 = \sum_{i=1}^{n} d_i, \quad T_2 = 2m + \sum_{i=1}^{n} d_i^2, \quad T_3 = 6(t^- - t^+) + 3\sum_{i=1}^{n} d_i^2 + \sum_{i=1}^{n} d_i^3.$$

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Proof (Formula for T_3). We have

$$T_3 = \mathrm{tr}\,(D - A)^3 = \mathrm{tr}\,D^3 + 3\mathrm{tr}\,A^2 D - 3\mathrm{tr}\,AD^2 - \mathrm{tr}\,A^3.$$

From $\mathrm{tr}\,AD^2 = 0$, $\mathrm{tr}\,A^2 D = \sum_{i=1}^{n} d_i^2$, and $\mathrm{tr}\,(A^3) = 6(t^+ - t^-)$, we get the assertion.

A general result

From the first three Laplacian spectral moments we deduce:

Theorem

Let $\Gamma = (G, \sigma) \sim \Sigma = (H, \sigma')$. *Then,*

(i) Γ *and* Σ *have the same number of vertices and edges;*

(ii) Γ *and* Σ *have the same number of balanced components;*

(iii) Γ *and* Σ *have the same Laplacian spectral moments;*

(iv) Γ *and* Σ *have the same sum of squares of degrees;*

(v) $6(t_\Gamma^- - t_\Gamma^+) + \sum_{i=1}^n d_G(v_i)^3 = 6(t_\Sigma^- - t_\Sigma^+) + \sum_{i=1}^n d_H(v_i)^3$.

Interlacing Theorem

The following result is the interlacing theorem in the edge variant. It can be deduced from the ordinary vertex variant interlacing theorem for the adjacency matrix.

Theorem

Let $\Gamma = (G, \sigma)$ be a signed graph and $\Gamma - e$ be the signed graph obtained from Γ by deleting the edge e. Then

$$\mu_1(\Gamma) \geq \mu_1(\Gamma - e) \geq \mu_2(\Gamma) \geq \mu_2(\Gamma - e) \geq \ldots \geq \mu_n(\Gamma) \geq \mu_n(\Gamma - e)$$

Some work done and some possible investigations

Some spectral determination/characterization problems already investigated:

- Laplacian Spectral determination of friendship graphs (Francesco Belardo, J.F. Wang);
- Laplacian Spectral determination of lollipop graphs (with Francesco Belardo);

There are many interesting classes that still have not been investigated:
Signed graphs with small largest eigenvalue, signed bicyclic graphs, paths and cycles...

Signed lollipop graphs

The theory of signed Trees degenerates to the that of unsigned trees since all trees on the same underlying graph are switching equivalent. So the first non-trivial case is to consider unicyclic graphs. So, we now consider the signed lollipop graphs.

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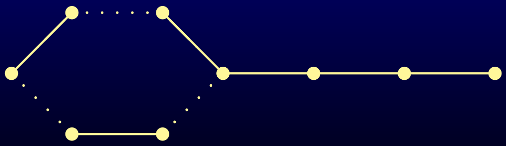
The theory of signed Trees degenerates to the that of unsigned trees since all trees on the same underlying graph are switching equivalent. So the first non-trivial case is to consider unicyclic graphs. So, we now consider the signed lollipop graphs.

By $L_{g,n}$ we denote the lollipop graph whose girth is g and the order is n. Since the lollipop is a unicyclic graph, then it admits only two different non-equivalent signatures: the all positive edges $\sigma = +$, and the one which makes the cycle unbalanced that will be denoted by $\bar{\sigma}$.



The signed lollipop graph $(L_{6,9}, \bar{\sigma})$.

Algebraic properties of the signed lollipop graph

For a signed graph Γ, $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq 0$ are the L-eigenvalues.

Lemma

Let $(L_{g,n}, \sigma)$ be a lollipop graph. Then we have

$$5 \; > \; \mu_1(L_{g,n}, \sigma) \; > \; 4 > \; \mu_2(L_{g,n}, \sigma).$$

Proof Since $\mu_1(\Gamma) < \Delta_1 + \Delta_2$, and the largest vertex degrees of $(L_{g,n}, \sigma)$ are 3 and 2, we obtain $\mu_1(L_{g,n}, \sigma) < 5$.

Since the path P_n is an edge-deleted subgraph, by IT

$$\mu_1(L_{g,n}, \sigma) \geq 4 > \mu_1(P_n, \sigma) \geq \mu_2(L_{g,n}, \sigma).$$

Finally, it is possible to prove that 4 cannot be an eigenvalue since $\psi((L_{g,n}, \sigma), 4) \neq 0$.

The structure of the cospectral mates

Lemma

Let $\Gamma \sim (L_{g,n}, \sigma)$, then Γ has the degree sequence of $(L_{g,n}, \sigma)$.

From $\mu_1(L_{g,n}, \sigma) < 5$, it is $\Delta(\Gamma) < 4$, otherwise $K_{1,4} \subseteq \Gamma$ and, by IT, we get $\mu_1(\Gamma) \geq 5$.

Let n_i be the number of vertices whose degree is i, where $0 \leq i \leq 3$. From the spectral moments we deduce:

$$\begin{cases} n_0 + n_1 + n_2 + n_3 & = & n \\ n_1 + 2n_2 + 3n_3 & = & 2n \\ n_1 + 4n_2 + 9n_3 & = & 4n + 2. \end{cases}$$

The unique solution is $n_0 = 0$, $n_1 = 1$, $n_2 = n - 2$ and $n_3 = 1$. So Γ consists of a lollipop graph with possibly one or more cycles.

L-spectra of paths and cycles

Lemma

Let P_n and C_n be the path and the cycle on n vertices, respectively. Let $\mathrm{Spec}_L(\Gamma)$ denote the multiset of eigenvalues of $L(\Gamma)$.

$$\mathrm{Spec}_L(C_{2n},+) = \{2+2\cos\frac{2k}{2n}\pi,\ k=0,1,\dots,2n-1\};$$
$$\mathrm{Spec}_L(C_{2n+1},+) = \{2+2\cos\frac{2k+1}{2n+1}\pi,\ k=0,1,\dots,2n\};$$
$$\mathrm{Spec}_L(C_{2n},\bar{\sigma}) = \{2+2\cos\frac{2k+1}{2n}\pi,\ k=0,1,\dots,2n-1\};$$
$$\mathrm{Spec}_L(C_{2n+1},\bar{\sigma}) = \{2+2\cos\frac{2k}{2n+1}\pi,\ k=0,1,\dots,2n\};$$
$$\mathrm{Spec}_L(P_n) = \{2+2\cos\frac{k}{n}\pi,\ k=1,2,\dots,n\}.$$

L-spectra of cycles

From the previos lemma we get that

- $(C_{2n}, +) \sim (C_n, +) \cup (C_n, \bar{\sigma})$;
- $\mathrm{Spec}_L\,(C_{2n+1}, +) \supseteq \mathrm{Spec}_L\,(C_d, +)$ for any d divisor of $2n+1$;
- $\mathrm{Spec}_L\,(C_{2n+1}, \bar{\sigma}) \supseteq \mathrm{Spec}_L\,(C_d, \bar{\sigma})$ for any d divisor of $2n+1$;
- $\mathrm{Spec}_L\,(C_{2n+1}, \bar{\sigma}) \supseteq \mathrm{Spec}_L\,(C_d, \bar{\sigma})$ provided that $\frac{2n}{d}$ is odd;
- $\mathrm{Spec}_L\,(C_{2n}, +) \supseteq \mathrm{Spec}_L\,(C_d, +)$ provided that $\frac{2n}{d}$ is even.

The lemma below follows from the above observations.

Lemma

Let $(C_{2n}, +)$ be a even balanced cycle and let $2n = 2^{t+1}r$, where t and r are positive integer and r is odd. For $0 \leq s \leq t$,

- *If $r \geq 3$, then $(C_{2^{t+1}r}, +) \sim (C_{2^s r}, +) \bigcup_{i=s}^{t} (C_{2^i r}, \bar{\sigma})$;*
- *If $r = 1$ then $(C_{2^{t+1}}, +) \sim (C_{2^s}, +) \bigcup_{i=s}^{t} (C_{2^i}, \bar{\sigma})$.*

The cycles have eigenvalues of multiplicity two. Has the lollipop multiple eigenvalues?

Theorem

The signed lollipop graph $(L_{g,n}, \sigma) = \Lambda$ *has simple L-eigenvalues if* $\mathrm{GCD}(g, n) = 1$.
If $\mathrm{GCD}(g, n) = d \geq 2$*, then we have:*

- *if* g *is odd, then the eigenvalues of* Λ *of multiplicity two are those of* (C_d, σ)*;*
- *if* g *is even,* $\frac{d}{g}$ *odd (resp., even), and* $\sigma = +$*, then the eigenvalues of* Λ *of multiplicity two are those of* $(C_d, +)$ *(resp.,* $(C_{2d}, +)$*);*
- *if* g *is even and* $\sigma = \bar{\sigma}$*, then for* $\frac{g}{d}$ *odd the eigenvalues of* Λ *of multiplicity two are those of* $(C_d, \bar{\sigma})$*, while for* $\frac{g}{d}$ *even,* Λ *has just simple eigenvalues.*

Proof (sketch)

Let μ be, if any, an eigenvalue of multiplicity two, then μ is an L-eigenvalue of $(C_g, \sigma) \cup P_{n-g}$ (by IT). If we consider the subdivision $\sqrt{\mu} = \lambda$ is an eigenvalue of $\mathcal{S}(L_{g,n}, \sigma) = (L_{2g,2n}, \sigma')$.

By decomposing the characteristic polynomial at the pendant path edge incident the vertex of degree 3 we have:

$$\phi(L_{2g,2n}, \sigma') = \phi(C_{2g}, \sigma')\phi(P_{2n-2g}) - \phi(P_{2g-1})\phi(P_{2n-2g-1}).$$

Since λ is a root of $\phi(C_{2g}, \sigma')$ or a root of $\phi(P_{2n-2g-1})$, then λ is of multiplicity two if and only if λ is a root of both $\phi(C_{2g}, \sigma')$ and $\phi(P_{2n-2g-1})$, that is, μ is an eigenvalue of both (C_g, σ) and P_{n-g}.

The claim follows by analyzing when (C_g, σ) and P_{n-g} share some eigenvalue.

Final results

After come tricky lemmas (which I will not propose you), we finally are able to prove that the cospectral mate is connected...

Lemma

If $\Gamma \sim (L_{g,n}, \sigma)$ then Γ is a signed lollipop graph.

So it remains to check whether two non switching isomorphic lollipop graphs can be cospectral. But the answer is negative:

Theorem

No two non switching isomorphic lollipop graphs can be L-cospectral.

Proof (Sketch)

By Shwenks's formulas it is possible to decompose the polynomial of the lollipop in terms of paths. Let $\varsigma = (-1)^g \sigma(\Lambda)$.

$$\begin{aligned}\psi(\Lambda, x) &= \frac{1}{x}(\psi(P_{n-g+1}) + \psi(P_{n-g}))\Big[\frac{x-3}{x}\psi(P_g) - \frac{2}{x}\psi(P_{g-1}) \\ &\quad +2\varsigma\Big] - \frac{1}{x^2}\psi(P_g)(\psi(P_{n-g}) + \psi(P_{n-g-1})).\end{aligned}$$

The formula $\psi(P_n) = (x-2)\psi(P_{n-1}) - \psi(P_{n-2})$ can be seen as a homogeneous second order recurrence equation

$$p_n = (x-2)p_{n-1} - p_{n-2},$$

with $p_0 = 0$ and $p_1 = x$ as boundary conditions.

Proof (continuation)

It is a matter of computation to check that the solution is

$$p_n = \frac{(y^{2n} - 1)(y+1)}{y^n(y-1)},$$

where y satisfies the characteristic equation $y^2 - (x-2)y + 1 = 0$. Let

$$\Phi(\Gamma) = y^n\,(y-1)^2\,\psi(\Gamma, y) - (y^{2n+2} - 2y^{2n+1} - 2y + 1),$$

then, by applying the above described transformation, we get

$$\Phi(L_{g,n}, \sigma) = 2\varsigma y^{2n-g+2} - 2\varsigma y^{2n-g+1} + y^{2n-2g+2} + y^{2g} - 2\varsigma y^{g+1} + 2\varsigma y^{g}$$

From the above polynomial, two signed lollipop are L-cospectral iff both g and $\sigma(\Lambda)$ are the same, i.e., they are switching isomorphic.

Thank you!