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On the eigenspaces of signed line graphs and signed subdivision graphs

Outline

1. Preliminaries
   - Basic notions on Signed Graphs
   - Matrices of Signed Graphs
2. Relations between spectra
   - Signed graphs, Bi-directed graphs and Mixed graphs
   - Relations among the eigenvalues
3. Relations among the eigenspaces
   - Eigenspaces of the signed line graph
   - Eigenspaces of the signed subdivision graph

Signed Graphs

A *signed graph* Γ is an ordered pair (G, σ), where $G = (V(G), E(G))$ is a graph and $\sigma : E(G) \to \{+, -\}$ is the signature function (or sign mapping) on the edges of G.

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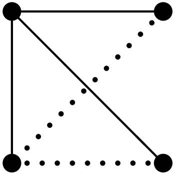

Example of a signed graph.

positive edges = solid lines;
negative edges = dotted lines.

More on Signed Graphs

Signed graphs were first introduced by Harary to handle a problem in social psychology (Cartwright and Harary, 1956). Recently, signed graphs have been considered in the study of complex networks, and Godsil et al. showed that negative edges are useful for perfect state transfer in quantum computing.

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In most applications of signed graphs there is a recurring property that naturally arises:

Definition

A signed graph is said to be balanced if and only if all its cycles are positive.

Balance

The first characterization of balance is due to Harary:

Theorem (Harary, 1953)

A signed graph is balanced iff its vertex set can be divided into two sets (either of which may be empty), so that each edge between the sets is negative and each edge within a set is positive.

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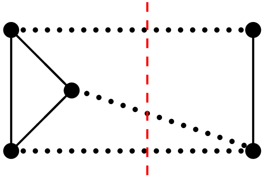
The above theorem shows that balancedness is a generalization of the ordinary bipartiteness in (unsigned) graphs.



A balanced signed graph.
The dashed line separates
the two clusters.

Signature Switching

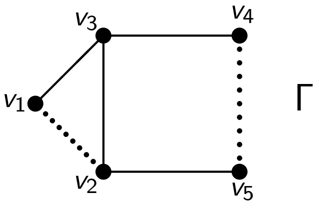
Definition

Let $\Gamma = (G, \sigma)$ *be a signed graph and* $U \subseteq V(G)$*. The signed graph* Γ^U *obtained by negating the edges in the cut* $[U; U^c]$ *is a* (sign) switching *of* Γ*. We also say that the signatures of* Γ^U *and* Γ *are* equivalent*.*

The signature switching preserves the set of the positive cycles.

In general, we say that two signed graphs are *switching isomorphic* if their underlying graphs are isomorphic and the signatures are switching equivalent. The set of signed graphs switching isomorphic to Γ is the *switching isomorphism class* of Γ, written $[\Gamma]$.

Example of switching equivalent graphs



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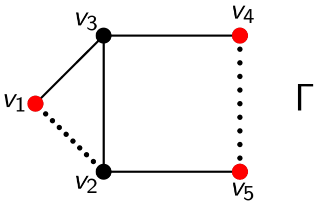

Let $U = \{v_1, v_4, v_5\}$.

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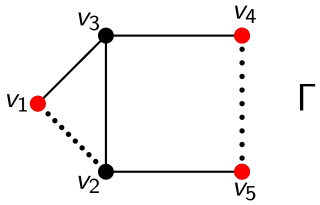

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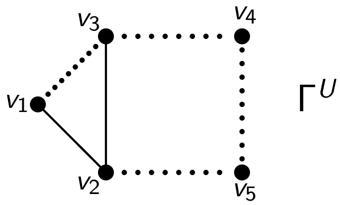

Note that the switching preserves the sign of the cycles!

Matrices of (unsigned) graphs

Let $M = M(G)$ be a *graph matrix* defined in a prescribed way. The *M-polynomial* of G is defined as $\det(\lambda I - M)$, where I is the identity matrix. The *M-spectrum* of G is a multiset consisting of the eigenvalues of $M(G)$. The largest eigenvalue of $M(G)$ is called the *M-spectral radius* of G.

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Some well-known graph matrices of a (unsigned) graph G are:

- the adjacency matrix $A(G)$;
- the Laplacian matrix $L(G) = D(G) - A(G)$;
- the signless Laplacian matrix $Q(G) = D(G) + A(G)$;
- their normalized variants.

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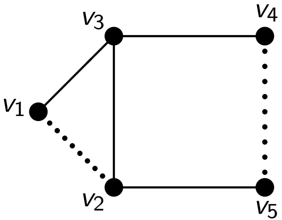
The adjacency matrix and the Laplacian matrix (and normalized variants) can be similarly defined for signed graphs.

Adjacency matrix of Signed Graphs

The adjacency matrix is defined as $A(\Gamma) = (a_{ij})$, where

$$a_{ij} = \begin{cases} \sigma(v_i v_j), & \text{if } v_i \sim v_j; \\ 0, & \text{if } v_i \not\sim v_j. \end{cases}$$

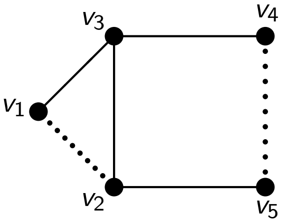


$$\rightsquigarrow \quad A(\Gamma) = \begin{pmatrix} 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 \end{pmatrix}$$

Laplacian of Signed Graphs

The Laplacian matrix of $\Gamma = (G, \sigma)$ is defined as $L(\Gamma) = D(G) - A(\Gamma) = (l_{ij})$

$$l_{ij} = \begin{cases} \deg(v_i), & \text{if } i = j; \\ -\sigma(v_i v_j), & \text{if } i \neq j. \end{cases}$$



$$\rightsquigarrow \quad L(\Gamma) = \begin{pmatrix} 2 & 1 & -1 & 0 & 0 \\ 1 & 3 & -1 & 0 & -1 \\ -1 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 2 & 1 \\ 0 & -1 & 0 & 1 & 2 \end{pmatrix}$$

Switching and signature similarity

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Switching has a matrix counterpart. In fact, Let Γ and $\Gamma' = \Gamma^U$ be two switching equivalent graphs.
Consider the matrix $S_U = \mathrm{diag}(s_1, s_2, \ldots, s_n)$ such that

$$s_i = \begin{cases} +1, & i \in U \\ -1, & i \in \Gamma \setminus U \end{cases}$$

S_U is called a *signature matrix* (or *state* matrix).

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S_U is called a *signature matrix* (or *state* matrix).

It is easy to check that

$$A(\Gamma^U) = S_U \; A(\Gamma) \; S_U \quad \text{and} \quad L(\Gamma^U) = S_U \; L(\Gamma) \; S_U.$$

Hence, signed graphs from the same switching class share similar graph matrices, or switching isomorphic graphs are cospectral.

Balance and signature switching

The following theorem is pretty evident:

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Proof. If the graph is balanced then it admits a bipartition in a 2-clusters, so we can switch all the negative edges to positive edges. On the other hand, if the signed graph is switching equivalent to the all positive signature then all cycles are balanced and then the whole graph is balanced as well. □

A signed graph that is switching equivalent to the all negative signature is said to be *antibalanced*.

(signless) Laplacian spectral theory of unsigned graphs

If the signed graph $\Gamma = (G, \sigma)$ has only:

(signless) Laplacian spectral theory of unsigned graphs

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$\swarrow$

positive edges

$$\sigma(e) = +1 \text{ for all } e \in E(G)$$

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we have the signless Laplacian matrix of G.

(signless) Laplacian spectral theory of unsigned graphs

If the signed graph $\Gamma = (G, \sigma)$ has only:

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positive edges	**negative** edges
$\sigma(e) = +1$ for all $e \in E(G)$	$\sigma(e) = -1$ for all $e \in E(G)$
$A(\Gamma) = A(G)$	$A(\Gamma) = -A(G)$
$L(\Gamma, +) = L(G)$	$L(\Gamma, -) = Q(G)$
we have the usual Laplacian matrix of G.	we have the signless Laplacian matrix of G.

The Laplacian Theory of signed Graphs can be seen as a generalization of those of unsigned graphs.

Bi-directed graphs and signed graphs

An oriented signed graph is an ordered pair $\Gamma_\eta = (\Gamma, \eta)$, where

$$\eta : V(G) \times E(G) \to \{-1, +1, 0\}$$

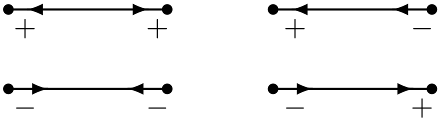
satisfies the following three conditions:

(*i*) $\eta(u, vw) = 0$ whenever $u \neq v, w$;

(*ii*) $\eta(v, vw) = +1$ (or -1) if an arrow at v is going into (resp. out of) v;

(*iii*) $\eta(v, vw)\eta(w, vw) = -\sigma(vw)$.

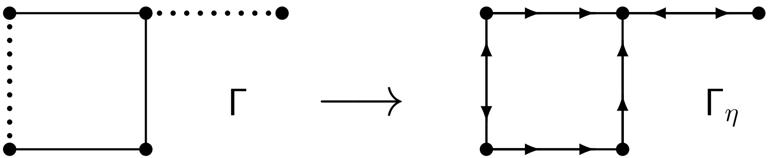
unoriented edges: oriented edges:



Bidirected edges

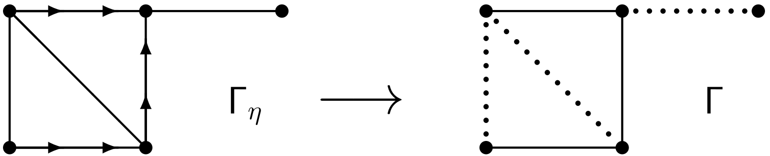
Bi-directed graphs and signed graphs

So we have that positive edges are oriented edges, while negative corresponds to unoriented. Thus each bi-directed graph is a signed graph. The converse is also true, but then one arrow (at any end) can be taken arbitrarily, while not the other arrow (in view of (iii)).



Mixed graphs and signed graphs

A mixed graph is a graph in which the edges can be either oriented or unoriented. Clearly, a mixed graph can be interpreted as a bi-directed graph. Consequently, mixed graphs can be treated as signed graphs, where the unoriented edges are negative edges and oriented edges are positive edges.



In the literature some results have been proved more than once due to the above fact!

Incidence matrix

The *incidence matrix* of Γ_η is the matrix $B_\eta = (b_{ij})$, whose rows correspond to vertices and columns to edges of G, such that

$$b_{ij} = \eta(v_i, e_j),$$

with $v_i \in V(G)$ and $e_j \in E(G)$. So each row of the incidence matrix corresponding to vertex v_i contains $deg(v_i)$ non-zero entries, each equal to $+1$ or -1. On the other hand, each column of the incidence matrix corresponding to edge e_j contains two non-zero entries, each equal to $+1$ or -1.

$$B_\eta B_\eta^T = D(G) - A(\Gamma) = L(\Gamma),$$

where $D(G)$ is the diagonal matrix of vertex degrees of G. So $L(\Gamma)$ is positive-semidefinite.
Note any choice for η leads to the same matrix $L(\Gamma)$!

Signed line graphs

On the other hand,

$$B_\eta^T B_\eta = 2I + A(\mathcal{L}(\Gamma_\eta)),$$

where $\mathcal{L}(\Gamma_\eta)$ is a signed graph whose underlying graph is $\mathcal{L}(G)$.

The signed line graph of $\Gamma = (G, \sigma)$ is the signed graph $(\mathcal{L}(G), \sigma_\eta^{\mathcal{L}})$, where $\mathcal{L}(G)$ is the (usual) line graph and

$$\sigma_\eta^{\mathcal{L}}(e_i e_j) = \begin{cases} b_{ki}^\eta b_{kj}^\eta & \text{if } e_i \text{ is incident } e_j \text{ at } v_k; \\ 0 & \text{otherwise.} \end{cases}$$

Assigning a different orientation η' will lead to a different $\mathcal{L}(\Gamma_{\eta'})$, however we have that $\mathcal{L}(\Gamma_\eta)$ and $\mathcal{L}(\Gamma_{\eta'})$ are switching equivalent!

Signed line graphs

So the signed line graph $\mathcal{L}(\Gamma)$ is uniquely defined up to switching isomorphisms. On the other hand,

$$B_\eta B_\eta^T = L(\Gamma), \quad \text{and} \quad B_\eta^T B_\eta = \mathcal{L}(\Gamma)$$

share the same non-zero eigenvalues, and we have the following theorem:

Theorem

Let Γ be a signed graph of order n and size m, and let $\phi(\Gamma)$ and $\psi(\Gamma)$ be its adjacency and Laplacian characteristic polynomials, respectively. Then it holds

$$\phi(\mathcal{L}(\Gamma), x) = (x+2)^{m-n}\psi(\Gamma, x+2).$$

Signed subdivision graphs

The signed subdivision graph, associated to B_η, is the signed graph $\mathcal{S}(\Gamma_\eta) = (\mathcal{S}(G), \sigma_\eta^{\mathcal{S}})$, where $\mathcal{S}(G)$ is the usual subdivision of unsigned graphs and

$$\sigma_\eta^{\mathcal{S}}(v_i e_j) = b_{ij}^\eta$$

In matrix form (O_t is the $t \times t$ zero matrix):

$$A(\mathcal{S}(\Gamma_\eta)) = \begin{pmatrix} O_n & B_\eta \\ B_\eta^\top & O_m \end{pmatrix}.$$

Theorem

Let Γ be a signed graph of order n and size m, and $\phi(\Gamma)$ and $\psi(\Gamma)$ its adjacency and Laplacian polynomials, respectively. Then

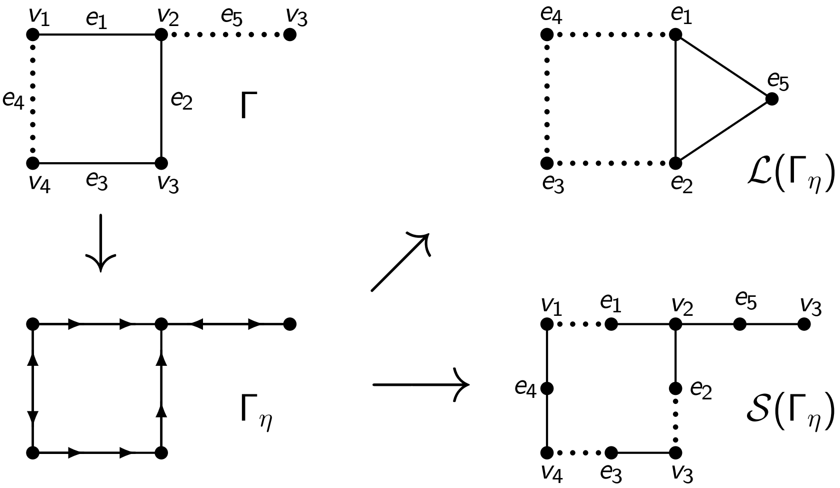
$$\phi(\mathcal{S}(\Gamma), x) = x^{m-n}\psi(\Gamma, x^2).$$

Proof

Recall that $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |D||A - BD^{-1}C|$. Hence,

$$\begin{aligned}
\phi(\mathcal{S}(\Gamma), x) &= \begin{vmatrix} xI_n & -B \\ -B^\top & xI_m \end{vmatrix} = x^m|(xI_n) - B(xI_m)^{-1}B^\top| = \\
&= x^m|(xI_n) - \frac{1}{x}BB^\top| \\
&= x^m|\frac{1}{x}(x^2I_n - BB^\top)| \\
&= x^{m-n}|x^2I_n - L(\Gamma)| \\
&= x^{m-n}\,\psi(\Gamma, x^2). \quad \square
\end{aligned}$$

An example



A signed graph and the corresponding signed subdivision and line graph.

Relations among the spectra

The previous facts can be synthesized by the following theorem:

Theorem

Let Γ be a signed graph of order n and size m, and $\phi(\Gamma)$ and $\psi(\Gamma)$ its adjacency and Laplacian polynomials, respectively. Then

(i) $\phi(\mathcal{L}(\Gamma), x) = (x+2)^{m-n}\psi(\Gamma, x+2)$,

(ii) $\phi(\mathcal{S}(\Gamma), x) = x^{m-n}\psi(\Gamma, x^2)$,

where $\mathcal{L}(\Gamma)$ and $\mathcal{S}(\Gamma)$ are the signed line graph and the signed subdivision graph of Γ, respectively.

What we can say about the eigenvectors of corresponding eigenvalues?

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Note: In the reminder Γ is connected.

Notation

We first focus our attention at the eigenvectors of the signed graph Γ and the corresponding signed line graph $\mathcal{L}(\Gamma_\eta)$.

We denote the L-eigenvalues of Γ by

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq 0,$$

and the A-eigenvalues of $\mathcal{L}(\Gamma)$ by

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq -2.$$

Then

$$\mu_i = \lambda_i + 2$$

for $i = 1, 2, \ldots, \min\{m, n\}$; for $i > \min\{m, n\}$ we have either $\mu_i = 0$ (if any) or $\lambda_i = -2$ (if any).

We now consider the first formula:

$$\phi(\mathcal{L}(\Gamma), x) = (x+2)^{m-n}\psi(\Gamma, x+2).$$

Let $\mathcal{E}_M(\nu;\Gamma)$ be the eigenspace of Γ related to the eigenvalue ν of the matrix $M = M(\Gamma)$.

Assume first that $\mu = \lambda + 2 \neq 0$, so it is $\lambda \neq -2$.

We have the two following claims:

Claim 1: If $\mathbf{x} \in \mathcal{E}_L(\mu;\Gamma) \setminus \{\mathbf{0}\}$, then $\mathbf{y} = B^\top \mathbf{x} \in \mathcal{E}_A(\lambda;\mathcal{L}(\Gamma)) \setminus \{\mathbf{0}\}$;

Claim 2: If $\mathbf{y} \in \mathcal{E}_A(\lambda;\mathcal{L}(\Gamma)) \setminus \{\mathbf{0}\}$, then $\mathbf{x} = B\mathbf{y} \in \mathcal{E}_L(\mu;\Gamma) \setminus \{\mathbf{0}\}$.

From the above claims, if $\mu \neq 0$, or equivalently if $\lambda \neq -2$, we have that the above two vector spaces are isomorphic.

Proof of Claim 1

Recall that

$$BB^\top = D(G) - A(\Gamma) = L(\Gamma), \quad B^\top B = 2I + A(\mathcal{L}(\Gamma)). \tag{1}$$

Let $\mathbf{x}$ be a μ-eigenvector of Γ, that is $L\mathbf{x} = BB^\top\mathbf{x} = \mu\mathbf{x}$ and let $\mathbf{y} = B^\top\mathbf{x}$. Hence we obtain $\mu\mathbf{x} = B\mathbf{y}$.

Clearly, $\mathbf{x} \in \boldsymbol{R}^n$ and $\mathbf{y} \in \boldsymbol{R}^m$, and both are non-zero vectors.

Next we have that

$$B^\top B\mathbf{y} = B^\top BB^\top\mathbf{x} = B^\top L\mathbf{x} = \mu B^\top\mathbf{x} = \mu\mathbf{y}.$$

Therefore, by the second equality in (1), we have

$$A_{\mathcal{L}}\mathbf{y} = (\mu - 2)\mathbf{y} = \lambda\mathbf{y}.$$

Hence, $\mathbf{y} = B^\top\mathbf{x} \neq \mathbf{0}$ is a $(\mu - 2)$-eigenvector of $\mathcal{L}(G)$. $\square$

The eigenspace $\mathcal{E}_L(0;\Gamma)$

The following fact is well-known:

Lemma

Let Γ be a signed graph. Then, Γ is balanced iff $\mu_n(\Gamma) = 0$.

From the characterization given by Harary, $\Gamma = (G, \sigma)$ is balanced if and only if the vertex set is partioned in two color classes such that negative edges appear only between the two classes and positive edges appear only within the classes.

The L-eigenvector $\mathbf{x}$ related to $\mu = 0$ has entries -1 for vertices from one color class, while $+1$ otherwise. But then the corresponding vector $\mathbf{y} = B^\top \mathbf{x}$ is equal to $\mathbf{0}$, and so $\mathbf{y}$ is not an eigenvector for $\mathcal{L}(\Gamma)$.

Namely, $\mathcal{E}_L(0;\Gamma)$ and $\mathcal{E}_A(-2;\mathcal{L}(\Gamma))$ are not related.

The eigenspace $\mathcal{E}_A(-2; \mathcal{L}(\Gamma))$

The A-eigenspace of $\mathcal{L}(\Gamma)$ for $\lambda = -2$ can be obtained by using the *Star complement technique* from the theory of Hermitian matrices. From the following result, we get the star complements for -2:

Lemma

If Γ is a connected signed graph on m edges then

$$(-1)^m \phi(\mathcal{L}(\Gamma), -2) = \begin{cases} m+1 & \text{if } \Gamma \text{ is a tree,} \\ 4 & \text{if } \Gamma \text{ is an unbal. unicyclic graph,} \\ 0 & \text{otherwise.} \end{cases}$$

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Corollary

Let Γ be a connected signed graph. Then -2 is the least eigenvalue of $\mathcal{L}(\Gamma)$ if and only if Γ contains as a signed subgraph at least one balanced cycle, or at least two unbalanced cycles.

Eigenvectors for $\lambda = -2$

Similarly to the case of unsigned graphs, $\mathcal{E}_A(-2; \mathcal{L}(\Gamma))$ is directly obtained from two kinds of spanning subgraphs of Γ: the balanced cycles and/or the double-unbalanced dumbbells.

Theorem (Balanced cycle)

Let Θ be a balanced cycle and Θ_L its line (signed) graph. Then the vector $\mathbf{a} = (a_0, a_1, \ldots, a_{q-1})^\top$, where

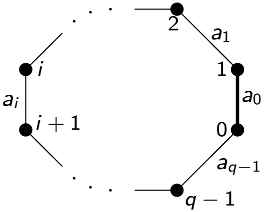
$$a_i = (-1)^i \left[\prod_{s=1}^{i} \nu(s) \right] a_0 \quad (i = 0, 1, \ldots, q-1),$$

where $\nu(s) = \sigma_L(e_{s-1}e_s)$, is an eigenvector of Θ_L for -2. Moreover, it can be extended to a (-2)-eigenvector of Γ_L by inserting zeros at remaining entries.

The eigenvector for $\lambda = -2$:

$$a_i = (-1)^i [\prod_{s=1}^{i} \nu(s)] a_0 \quad (i = 0, 1, \ldots, q-1),$$

where $\nu(j) = \sigma_L(e_{j-1}e_j)$.



In the case of double unbalanced dumbbell (two unbalanced cycles joined by a path), we have a similar result.

Main Result

Theorem

Let B_η be the incidence matrix of a connected sigraph Γ_η. Then

(i) *Let $\mu \neq 0$. $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_s}\}$ is an eigenbasis of $\mathcal{E}_L(\mu;\Gamma)$ if and only if $\{B^\top \mathbf{u_1}, B^\top \mathbf{u_2}, \ldots, B^\top \mathbf{u_s}\}$ is an eigenbasis of $\mathcal{E}_A(\mu - 2; \mathcal{L}(\Gamma))$;*

(ii) *Let $\lambda \neq -2$. $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_t}\}$ is an eigenbasis of $\mathcal{E}_A(\lambda; \mathcal{L}(\Gamma))$ if and only if $\{B\mathbf{v_1}, B\mathbf{v_2}, \ldots, B\mathbf{v_t}\}$ is a eigenbasis of $\mathcal{E}_L(\lambda + 2; \Gamma)$;*

(iii) *If $\mu = 0$, then Γ is balanced and $\mathcal{E}_L(0;\Gamma)$ is spanned by the vector $(x_1, x_2, \ldots, x_n)^\top$, where $x_i = -1$ in one color class and $x_i = +1$ otherwise;*

(iv) *If $\lambda = -2$, then the corresponding $\mathcal{E}_A(-2; \mathcal{L}(\Gamma))$ is spanned by the vectors constructed on the edges of balanced cycles and double-unbalanced dumbbells.*

Now we consider the formula

$$\phi(\mathcal{S}(\Gamma), x) = x^{m-n}\psi(\Gamma, x^2).$$

Let

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n, \quad \text{and} \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m,$$

be the L-eigenvalues of Γ, and the A-eigenvalues of $\mathcal{L}(\Gamma)$, respectively.

Observe that the A-eigenvalues of $\mathcal{S}(\Gamma)$ are $\pm\sqrt{\mu_i}$ ($i = 1, 2, \ldots, s$), for some $s \leq n$; all other eigenvalues are equal to 0. Note also that all non-zero L-eigenvalues of Γ and the corresponding A-eigenvalues $\mathcal{S}(\Gamma)$ have the same multiplicities. Also, if both μ_i and λ_i exist for some i, then $\mu_i = \lambda_i + 2$.

Relations among the matrices

Recall that the signed subdivision graph of Γ is the signed graph $\mathcal{S}(\Gamma)$ whose adjacency matrix is

$$A(\mathcal{S}(\Gamma)) = \begin{pmatrix} O_n & B \\ B^\top & O_m \end{pmatrix}.$$

Evidently,

$$\begin{aligned} A^2(\mathcal{S}(\Gamma)) &= \begin{pmatrix} BB^\top & O_n \\ O_m & B^\top B \end{pmatrix} = \begin{pmatrix} L(\Gamma) & O_n \\ O_m & A(\mathcal{L}(\Gamma)) + 2I_m \end{pmatrix} \\ &= L(\Gamma) \dot{+} (A(\mathcal{L}(\Gamma)) + 2I_m). \end{aligned}$$

The eigenvectors of $\mathcal{S}(\Gamma)$

Let $\hat{\lambda}$ be an A-eigenvalue of $\mathcal{S}(\Gamma)$ and $\mathbf{z}$ the corresponding eigenvector.

Without loss of generality, we can assume that the first n components of $\mathbf{z}$ correspond to the vertices, while the remaining m components to the edges of Γ. So we can write $\mathbf{z} = \mathbf{x} \dot{+} \mathbf{y}$. Clearly, $\mathbf{x} \in \boldsymbol{R}^n$ and $\mathbf{y} \in \boldsymbol{R}^m$.

Since $A(\mathcal{S}(\Gamma))\mathbf{z} = \hat{\lambda}\mathbf{z}$, then $A^2(\mathcal{S}(\Gamma)\mathbf{z} = \hat{\lambda}^2\mathbf{z}$. Since $A^2(\mathcal{S}(\Gamma)) = L(\Gamma) \dot{+} (A(\mathcal{L}(\Gamma)) + 2I_m)$, we obtain

$$L(\Gamma)\mathbf{x} = \hat{\lambda}^2\mathbf{x} \text{ and } (A(\mathcal{L}(\Gamma)) + 2I_m)\mathbf{y} = \hat{\lambda}^2\mathbf{y}.$$

Since $\mu = \hat{\lambda}^2$ and $\lambda = \mu - 2$ $(= \hat{\lambda}^2 - 2)$, we have

$$L(\Gamma)\mathbf{x} = \mu\mathbf{x} \text{ and } A(\mathcal{L}(\Gamma))\mathbf{y} = \lambda\mathbf{y}. \tag{2}$$

From $\mathcal{E}_A(\hat{\lambda}; \mathcal{S}(\Gamma))$ to $\mathcal{E}_L(\hat{\lambda}^2; \Gamma)$ and $\mathcal{E}_A(\hat{\lambda}^2 - 2; \mathcal{L}(\Gamma))$

Hence an eigenvector of $\mathcal{S}(\Gamma)$ corresponding to $\hat{\lambda}$ is the join of the eigenvectors of Γ and $\mathcal{L}(\Gamma)$ corresponding to $\mu = \hat{\lambda}^2$ and $\lambda = \hat{\lambda}^2 - 2$, respectively.

Let

$$\{\mathbf{u_1} = \mathbf{v_1} \dot{+} \mathbf{w_1}, \mathbf{u_2} = \mathbf{v_2} \dot{+} \mathbf{w_2}, \ldots, \mathbf{u_k} = \mathbf{v_k} \dot{+} \mathbf{w_k}\}$$

be the corresponding eigenbasis of $\mathcal{E}_A(\hat{\lambda}; \mathcal{S}(\Gamma))$. Then

If $\hat{\lambda} > 0$

- $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ is an eigenbasis for $\mathcal{E}_L(\hat{\lambda}^2; \Gamma)$;
- $\{\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_k}\}$ is an eigenbasis for $\mathcal{E}_A(\hat{\lambda}^2 - 2; \mathcal{L}(\Gamma))$.

If $\hat{\lambda} = 0$

- $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ spans $\mathcal{E}_L(0; \Gamma)$;
- $\{\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_k}\}$ spans $\mathcal{E}_A(-2; \mathcal{L}(\Gamma))$.

From $\mathcal{E}_L(\mu;\Gamma)$ and $\mathcal{E}_A(\mu-2;\mathcal{L}(\Gamma))$ to $\mathcal{E}_A(\pm\sqrt{\mu};\mathcal{S}(\Gamma))$

Similarly, let

$$\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$$

be an eigenbasis of $\mathcal{E}_L(\mu;\Gamma)$. Then

- $\hat{\lambda} = \pm\sqrt{\mu} \neq 0$.

  $$\{\mathbf{u_1} = \mathbf{v_1}\dot{+}(\pm B^\top \mathbf{v_1}),\ \mathbf{u_2} = \mathbf{v_2}\dot{+}(\pm B^\top \mathbf{v_2}), \ldots, \mathbf{u_k} = \mathbf{v_k}\dot{+}(\pm B^\top \mathbf{v_k})\}$$

  is an eigenbasis of $\mathcal{E}_A(\pm\sqrt{\mu};\mathcal{S}(\Gamma))$;
- $\hat{\lambda} = \pm\sqrt{\mu} = 0$. If Γ is unbalanced, let $< \mathbf{v_1} >= \mathcal{E}_\mathbf{L}(\mathbf{0};\mathbf{\Gamma})$; otherwise, let $\mathbf{v_1} = \mathbf{0}$. Let $\{\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_k}\}$ be an A-eigenbasis of $\mathcal{L}(\Gamma)$ for $\lambda = -2$. Then the following set

  $$\{\mathbf{v_1}\dot{+}\mathbf{0},\ \mathbf{0}\dot{+}\mathbf{w_1},\ \mathbf{0}\dot{+}\mathbf{w_2}, \ldots, \mathbf{0}\dot{+}\mathbf{w_k}\} \setminus \{\mathbf{0}\}$$

  spans $\mathcal{E}_A(0;\mathcal{S}(\Gamma))$.

Thank you!!