# 2-DISTANCE-BALANCED GRAPHS

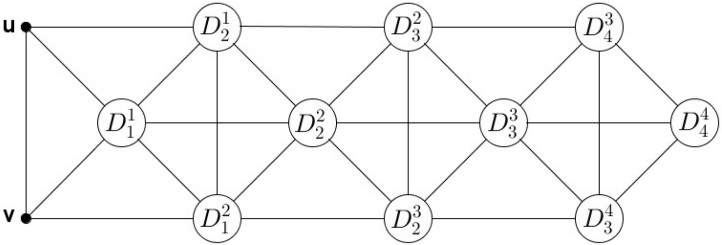
Boštjan Frelih
University of Primorska, Slovenia

Joint work with Štefko Miklavič

May 27, 2015

# Motivation: Distance-balanced graphs

A graph Γ is said to be distance-balanced if for any edge *uv* of Γ, the number of vertices closer to *u* than to *v* is equal to the number of vertices closer to *v* than to *u*.




A distance partition of a graph with diameter 4 with respect to edge *uv*.

# Motivation: Distance-balanced graphs

- These graphs were first studied (at least implicitly) by K. Handa who considered distance-balanced partial cubes.
- The term itself is due to J. Jerebic, S. Klavžar and D. F. Rall who studied distance-balanced graphs in the framework of various kinds of graph products.

# Motivation: Distance-balanced graphs

- These graphs were first studied (at least implicitly) by K. Handa who considered distance-balanced partial cubes.
- The term itself is due to J. Jerebic, S. Klavžar and D. F. Rall who studied distance-balanced graphs in the framework of various kinds of graph products.

# References


K. Handa, Bipartite graphs with balanced $(a, b)$-partitions, *Ars Combin.* **51** (1999), 113-119.

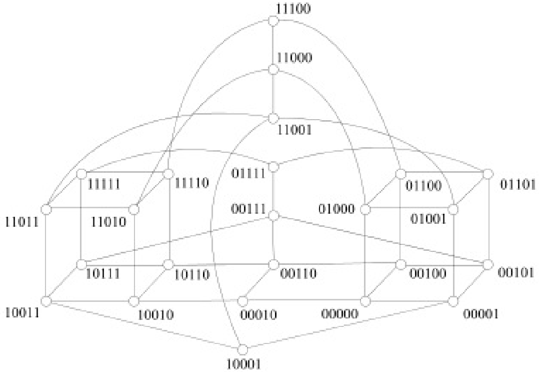
K. Kutnar, A. Malnič, D. Marušič, Š. Miklavič, Distance-balanced graphs: symmetry conditions, *Discrete Math.* **306** (2006), 1881-1894.

J. Jerebic, S. Klavžar, D. F. Rall, Distance-balanced graphs, *Ann. comb.* (Print. ed.) **12** (2008), 71-79.

# References


- K. Kutnar, A. Malnič, D. Marušič, Š. Miklavič, The strongly distance-balanced property of the generalized Petersen graphs. *Ars mathematica contemporanea*. (Print. ed.) **2** (2009), 41-47.
- A. Ilić, S. Klavžar, M. Milanović, On distance-balanced graphs, *Eur. j. comb.* **31** (2010), 733-737.
- Š. Miklavič, P. Šparl, On the connectivity of bipartite distance-balanced graphs, *Europ. j. Combin.* **33** (2012), 237-247.

# Distance-balanced graphs - example




Non-regular bipartite distance-balanced graph $H$.

# Generalization: $n$-distance-balanced graphs

A graph $\Gamma$ is said to be $n$-distance-balanced if there exist at least two vertices at distance $n$ in $\Gamma$ and if for any two vertices $u$ and $v$ of $\Gamma$ at distance $n$, the number of vertices closer to $u$ than to $v$ is equal to the number of vertices closer to $v$ than to $u$.

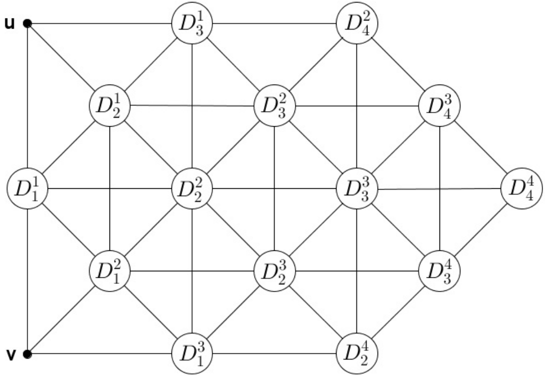
We are intrested in 2-distance-balanced graphs.

# Generalization: $n$-distance-balanced graphs

A graph $\Gamma$ is said to be $n$-distance-balanced if there exist at least two vertices at distance $n$ in $\Gamma$ and if for any two vertices $u$ and $v$ of $\Gamma$ at distance $n$, the number of vertices closer to $u$ than to $v$ is equal to the number of vertices closer to $v$ than to $u$.

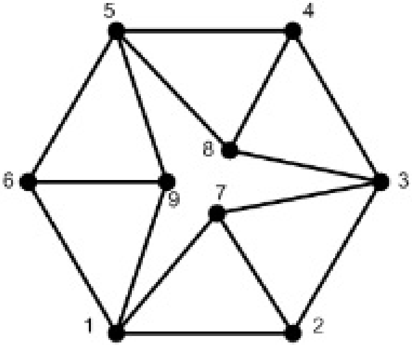
We are intrested in 2-distance-balanced graphs.

# 2-distance-balanced graphs




A distance partition of a graph with diameter 4 with respect to vertices $u$ and $v$ at distance 2.

# 2-distance-balanced graphs: example




Distance-balanced and 2-distance-balanced non-regular graph.

# 2-distance-balanced graphs

Question:

Are there any 2-distance-balanced graphs that are not distance-balanced?

Theorem (Handa, 1999):

Every distance-balanced graph is 2-connected.

A graph $\Gamma$ is $k$-vertex-connected (or k-connected) if it has more than $k$ vertices and the result of deleting any set of fewer than $k$ vertices is a connected graph.

Question:

Are there any connected 2-distance-balanced graphs that are not 2-connected?

# 2-distance-balanced graphs

**Question:**

Are there any 2-distance-balanced graphs that are not distance-balanced?

**Theorem (Handa, 1999):**

Every distance-balanced graph is 2-connected.

A graph $\Gamma$ is *$k$-vertex-connected* (or *k-connected*) if it has more than $k$ vertices and the result of deleting any set of fewer than $k$ vertices is a connected graph.

**Question:**

Are there any connected 2-distance-balanced graphs that are not 2-connected?

# 2-distance-balanced graphs

Question:

Are there any 2-distance-balanced graphs that are not distance-balanced?

Theorem (Handa, 1999):

Every distance-balanced graph is 2-connected.

A graph $\Gamma$ is *$k$-vertex-connected* (or k-connected) if it has more than $k$ vertices and the result of deleting any set of fewer than $k$ vertices is a connected graph.

Question:

Are there any connected 2-distance-balanced graphs that are not 2-connected?

# Family of graphs $\Gamma(G, c)$ - construction

Let $G$ be an arbitrary graph (not necessary connected) and $c$ an additional vertex.

Then $\Gamma(G, c)$ is a graph constructed in such a way that

$$V(\Gamma(G, c)) = V(G) \cup \{c\},$$

and

$$E(\Gamma(G, c)) = E(G) \cup \{cv \mid v \in V(G)\}.$$

- $\Gamma(G, c)$ is connected.
- $G$ is not connected $\iff \Gamma(G, c)$ is not 2-connected.
- Diameter of $\Gamma = \Gamma(G, c)$ is at most 2.

# Family of graphs $\Gamma(G, c)$ - construction

Let $G$ be an arbitrary graph (not necessary connected) and $c$ an additional vertex.

Then $\Gamma(G, c)$ is a graph constructed in such a way that

$$V(\Gamma(G, c)) = V(G) \cup \{c\},$$

and

$$E(\Gamma(G, c)) = E(G) \cup \{cv \mid v \in V(G)\}.$$

- $\Gamma(G, c)$ is connected.
- $G$ is not connected $\iff \Gamma(G, c)$ is not 2-connected.
- Diameter of $\Gamma = \Gamma(G, c)$ is at most 2.

# 2-distance-balanced graphs

**Question:**

Are there any connected 2-distance-balanced graphs that are not 2-connected?

**Theorem (B.F., Š. Miklavič)**

Graph $\Gamma$ is a connected 2-distance-balanced graph that is not 2-connected iff $\Gamma \cong \Gamma(G, c)$ for some not connected regular graph $G$.

# 2-distance-balanced graphs

### Question:

Are there any connected 2-distance-balanced graphs that are not 2-connected?

### Theorem (B.F., Š. Miklavič)

Graph $\Gamma$ is a connected 2-distance-balanced graph that is not 2-connected iff $\Gamma \cong \Gamma(G, c)$ for some not connected regular graph $G$.

# 2-distance-balanced graphs

### Lemma (B.F., Š. Miklavič)

If $G$ is regular but not a complete graph, then $\Gamma(G, c)$ is 2-distance-balanced.

Proof:

Let $G$ be a regular graph with valency $k$ and construct $\Gamma = \Gamma(G, c)$.

Let $G_1, G_2, \dots G_n$ be connected components of $G$ for some $n \geq 1$.

Two essentially different types of vertices at distance two in $\Gamma$:

1. both from $V(G_i)$,
2. one from $V(G_i)$, the other from $V(G_j)$ for $i \neq j$.

# 2-distance-balanced graphs

Lemma (B.F., Š. Miklavič)

If $G$ is regular but not a complete graph, then $\Gamma(G, c)$ is 2-distance-balanced.

Proof:

Let $G$ be a regular graph with valency $k$ and construct $\Gamma = \Gamma(G, c)$.

Let $G_1, G_2, \ldots G_n$ be connected components of $G$ for some $n \geq 1$.

Two essentially different types of vertices at distance two in $\Gamma$:

1. both from $V(G_i)$,
2. one from $V(G_i)$, the other from $V(G_j)$ for $i \neq j$.

# 2-distance-balanced graphs

## Lemma (B.F., Š. Miklavič)

If $G$ is regular but not a complete graph, then $\Gamma(G, c)$ is 2-distance-balanced.

Proof:

Let $G$ be a regular graph with valency $k$ and construct $\Gamma = \Gamma(G, c)$.

Let $G_1, G_2, \ldots G_n$ be connected components of $G$ for some $n \geq 1$.

Two essentially different types of vertices at distance two in $\Gamma$:

1. both from $V(G_i)$,
2. one from $V(G_i)$, the other from $V(G_j)$ for $i \neq j$.

# 2-distance-balanced graphs

Lemma (B.F., Š. Miklavič)

If $G$ is regular but not a complete graph, then $\Gamma(G, c)$ is 2-distance-balanced.

Proof:

Let $G$ be a regular graph with valency $k$ and construct $\Gamma = \Gamma(G, c)$.

Let $G_1, G_2, \ldots G_n$ be connected components of $G$ for some $n \geq 1$.

Two essentially different types of vertices at distance two in $\Gamma$:

1. both from $V(G_i)$,
2. one from $V(G_i)$, the other from $V(G_j)$ for $i \neq j$.

# Proof

1. Pick arbitrary $v_1, v_2 \in V(G_i)$ s.t. $\mathrm{d}(v_1, v_2) = 2$.

   $W^{\Gamma}_{v_1 v_2} = \{v_1\} \cup (N_{G_i}(v_1) \setminus (N_{G_i}(v_1) \cap N_{G_i}(v_2)))$

   $|W^{\Gamma}_{v_1 v_2}| = 1 + |N_{G_i}(v_1)| - |N_{G_i}(v_1) \cap N_{G_i}(v_2)|$

   $|W^{\Gamma}_{v_2 v_1}| = 1 + |N_{G_i}(v_2)| - |N_{G_i}(v_2) \cap N_{G_i}(v_1)|$

   $\Rightarrow |W^{\Gamma}_{v_1 v_2}| = |W^{\Gamma}_{v_2 v_1}|$

# Proof

1. Pick arbitrary $v_1, v_2 \in V(G_i)$ s.t. $\mathrm{d}(v_1, v_2) = 2$.

   $W^{\Gamma}_{v_1 v_2} = \{v_1\} \cup (N_{G_i}(v_1) \setminus (N_{G_i}(v_1) \cap N_{G_i}(v_2)))$

   $|W^{\Gamma}_{v_1 v_2}| = 1 + |N_{G_i}(v_1)| - |N_{G_i}(v_1) \cap N_{G_i}(v_2)|$

   $|W^{\Gamma}_{v_2 v_1}| = 1 + |N_{G_i}(v_2)| - |N_{G_i}(v_2) \cap N_{G_i}(v_1)|$

   $\Rightarrow |W^{\Gamma}_{v_1 v_2}| = |W^{\Gamma}_{v_2 v_1}|$

# Proof

1. Pick arbitrary $v_1, v_2 \in V(G_i)$ s.t. $\mathrm{d}(v_1, v_2) = 2$.

   $W^{\Gamma}_{v_1 v_2} = \{v_1\} \cup (N_{G_i}(v_1) \setminus (N_{G_i}(v_1) \cap N_{G_i}(v_2)))$

   $|W^{\Gamma}_{v_1 v_2}| = 1 + |N_{G_i}(v_1)| - |N_{G_i}(v_1) \cap N_{G_i}(v_2)|$

   $|W^{\Gamma}_{v_2 v_1}| = 1 + |N_{G_i}(v_2)| - |N_{G_i}(v_2) \cap N_{G_i}(v_1)|$

   $\Rightarrow |W^{\Gamma}_{v_1 v_2}| = |W^{\Gamma}_{v_2 v_1}|$

# Proof

1. Pick arbitrary $v_1, v_2 \in V(G_i)$ s.t. $\mathrm{d}(v_1, v_2) = 2$.

   $W^{\Gamma}_{v_1 v_2} = \{v_1\} \cup (N_{G_i}(v_1) \setminus (N_{G_i}(v_1) \cap N_{G_i}(v_2)))$

   $|W^{\Gamma}_{v_1 v_2}| = 1 + |N_{G_i}(v_1)| - |N_{G_i}(v_1) \cap N_{G_i}(v_2)|$

   $|W^{\Gamma}_{v_2 v_1}| = 1 + |N_{G_i}(v_2)| - |N_{G_i}(v_2) \cap N_{G_i}(v_1)|$

   $\Rightarrow |W^{\Gamma}_{v_1 v_2}| = |W^{\Gamma}_{v_2 v_1}|$

# Proof

1. Pick arbitrary $v_1, v_2 \in V(G_i)$ s.t. $\mathrm{d}(v_1, v_2) = 2$.

   $W^{\Gamma}_{v_1 v_2} = \{v_1\} \cup (N_{G_i}(v_1) \setminus (N_{G_i}(v_1) \cap N_{G_i}(v_2)))$

   $|W^{\Gamma}_{v_1 v_2}| = 1 + |N_{G_i}(v_1)| - |N_{G_i}(v_1) \cap N_{G_i}(v_2)|$

   $|W^{\Gamma}_{v_2 v_1}| = 1 + |N_{G_i}(v_2)| - |N_{G_i}(v_2) \cap N_{G_i}(v_1)|$

   $\Rightarrow |W^{\Gamma}_{v_1 v_2}| = |W^{\Gamma}_{v_2 v_1}|$

# Proof

2. Pick arbitrary $v \in V(G_i)$, $u \in V(G_j)$.

   $W_{vu}^{\Gamma} = \{v\} \cup N_{G_i}(v)$

   $W_{uv}^{\Gamma} = \{u\} \cup N_{G_j}(u)$

   $\Rightarrow |W_{vu}^{\Gamma}| = 1 + k = |W_{uv}^{\Gamma}|$

So $\Gamma = \Gamma(G, c)$ is 2-distance-balanced.

# Proof

2. Pick arbitrary $v \in V(G_i)$, $u \in V(G_j)$.

   $W^{\Gamma}_{vu} = \{v\} \cup N_{G_i}(v)$

   $W^{\Gamma}_{uv} = \{u\} \cup N_{G_j}(u)$

   $\Rightarrow |W^{\Gamma}_{vu}| = 1 + k = |W^{\Gamma}_{uv}|$

So $\Gamma = \Gamma(G, c)$ is 2-distance-balanced.

# Proof

2. Pick arbitrary $v \in V(G_i)$, $u \in V(G_j)$.

   $W^{\Gamma}_{vu} = \{v\} \cup N_{G_i}(v)$

   $W^{\Gamma}_{uv} = \{u\} \cup N_{G_j}(u)$

   $\Rightarrow |W^{\Gamma}_{vu}| = 1 + k = |W^{\Gamma}_{uv}|$

So $\Gamma = \Gamma(G, c)$ is 2-distance-balanced.

# Proof

2. Pick arbitrary $v \in V(G_i)$, $u \in V(G_j)$.

   $W_{vu}^{\Gamma} = \{v\} \cup N_{G_i}(v)$

   $W_{uv}^{\Gamma} = \{u\} \cup N_{G_j}(u)$

   $\Rightarrow |W_{vu}^{\Gamma}| = 1 + k = |W_{uv}^{\Gamma}|$

So $\Gamma = \Gamma(G, c)$ is 2-distance-balanced.

# Proof

2. Pick arbitrary $v \in V(G_i)$, $u \in V(G_j)$.

   $W^{\Gamma}_{vu} = \{v\} \cup N_{G_i}(v)$

   $W^{\Gamma}_{uv} = \{u\} \cup N_{G_j}(u)$

   $\Rightarrow |W^{\Gamma}_{vu}| = 1 + k = |W^{\Gamma}_{uv}|$

So $\Gamma = \Gamma(G, c)$ is 2-distance-balanced.

# 2-distance-balanced graphs

### Lemma (B.F., Š. Miklavič)

If $\Gamma$ is a connected 2-distance-balanced graph that is not 2-connected then $\Gamma \cong \Gamma(G, c)$ for some not connected regular graph $G$.

Proof:

Let $\Gamma$ connected 2-distance-balanced graph that is not 2-connected and $c$ a cut vertex in $\Gamma$.

If we delete vertex $c$, we get some subgraph $G$ with connected components $G_1, G_2, \ldots G_n$ for some $n \geq 2$.

# 2-distance-balanced graphs

### Lemma (B.F., Š. Miklavič)

If $\Gamma$ is a connected 2-distance-balanced graph that is not 2-connected then $\Gamma \cong \Gamma(G, c)$ for some not connected regular graph $G$.

Proof:

Let $\Gamma$ connected 2-distance-balanced graph that is not 2-connected and $c$ a cut vertex in $\Gamma$.

If we delete vertex $c$, we get some subgraph $G$ with connected components $G_1, G_2, \ldots G_n$ for some $n \geq 2$.

# 2-distance-balanced graphs

## Lemma (B.F., Š. Miklavič)

If $\Gamma$ is a connected 2-distance-balanced graph that is not 2-connected then $\Gamma \cong \Gamma(G, c)$ for some not connected regular graph $G$.

Proof:

Let $\Gamma$ connected 2-distance-balanced graph that is not 2-connected and $c$ a cut vertex in $\Gamma$.

If we delete vertex $c$, we get some subgraph $G$ with connected components $G_1, G_2, \ldots G_n$ for some $n \geq 2$.

# 2-distance-balanced graphs

### Lemma (B.F., Š. Miklavič)

If $\Gamma$ is a connected 2-distance-balanced graph that is not 2-connected then $\Gamma \cong \Gamma(G, c)$ for some not connected regular graph $G$.

Proof:

Let $\Gamma$ connected 2-distance-balanced graph that is not 2-connected and $c$ a cut vertex in $\Gamma$.

If we delete vertex $c$, we get some subgraph $G$ with connected components $G_1, G_2, \ldots G_n$ for some $n \geq 2$.

# Proof - Claim 1

Claim 1: $c$ is adjacent to every vertex in $G_l$ for at least one $l$, $1 \leq l \leq n$.

Suppose this statement is not true. Then for arbitrary $G_i$ and $G_j$:

$\exists v_2 \in V(G_i)$ s.t. $d(c, v_2) = 2$

$\Rightarrow \exists v_1 \in V(G_i)$ s.t. $d(c, v_1) = d(v_1, v_2) = 1$, and

$\exists u_2 \in V(G_j)$ s.t. $d(c, u) = 2$

$\Rightarrow \exists u_1 \in V(G_i)$ s.t. $d(c, u_1) = d(u_1, u_2) = 1$

# Proof - Claim 1

Claim 1: $c$ is adjacent to every vertex in $G_l$ for at least one $l$, $1 \leq l \leq n$.

Suppose this statement is not true. Then for arbitrary $G_i$ and $G_j$:

$\exists v_2 \in V(G_i)$ s.t. $d(c, v_2) = 2$

$\Rightarrow \exists v_1 \in V(G_i)$ s.t. $d(c, v_1) = d(v_1, v_2) = 1$, and

$\exists u_2 \in V(G_j)$ s.t. $d(c, u) = 2$

$\Rightarrow \exists u_1 \in V(G_i)$ s.t. $d(c, u_1) = d(u_1, u_2) = 1$

# Proof - Claim 1

$W^{\Gamma}_{cv_2} \supseteq \{c\} \cup V(G_j) \Rightarrow 1 + |V(G_j)| \leq |W^{\Gamma}_{cv_2}|$

$W^{\Gamma}_{v_2c} \subseteq V(G_i) \setminus \{v_1\} \Rightarrow |W^{\Gamma}_{v_2c}| \leq |V(G_i)| - 1$

$\Rightarrow |V(G_j)| \leq |V(G_i)| - 2$

$W^{\Gamma}_{cu_2} \supseteq \{c\} \cup V(G_i) \Rightarrow 1 + |V(G_i)| \leq |W^{\Gamma}_{cu_2}|$

$W^{\Gamma}_{u_2c} \subseteq V(G_j) \setminus \{u_1\} \Rightarrow |W^{\Gamma}_{u_2c}| \leq |V(G_j)| - 1$

$\Rightarrow |V(G_i)| + 2 \leq |V(G_j)|$

So: $|V(G_i)| + 2 \leq |V(G_j)| \leq |V(G_i)| - 2$, contradiction.

From now on (w.l.o.g.): $c$ is adjacent to every vertex in $V(G_1)$.

# Proof - Claim 1

$W^{\Gamma}_{cv_2} \supseteq \{c\} \cup V(G_j) \Rightarrow 1 + |V(G_j)| \leq |W^{\Gamma}_{cv_2}|$

$W^{\Gamma}_{v_2c} \subseteq V(G_i) \setminus \{v_1\} \Rightarrow |W^{\Gamma}_{v_2c}| \leq |V(G_i)| - 1$

$\Rightarrow |V(G_j)| \leq |V(G_i)| - 2$

$W^{\Gamma}_{cu_2} \supseteq \{c\} \cup V(G_i) \Rightarrow 1 + |V(G_i)| \leq |W^{\Gamma}_{cu_2}|$

$W^{\Gamma}_{u_2c} \subseteq V(G_j) \setminus \{u_1\} \Rightarrow |W^{\Gamma}_{u_2c}| \leq |V(G_j)| - 1$

$\Rightarrow |V(G_i)| + 2 \leq |V(G_j)|$

So: $|V(G_i)| + 2 \leq |V(G_j)| \leq |V(G_i)| - 2$, contradiction.

From now on (w.l.o.g.): $c$ is adjacent to every vertex in $V(G_1)$.

# Proof - Claim 1

$W^{\Gamma}_{cv_2} \supseteq \{c\} \cup V(G_j) \Rightarrow 1 + |V(G_j)| \leq |W^{\Gamma}_{cv_2}|$

$W^{\Gamma}_{v_2c} \subseteq V(G_i) \setminus \{v_1\} \Rightarrow |W^{\Gamma}_{v_2c}| \leq |V(G_i)| - 1$

$\Rightarrow |V(G_j)| \leq |V(G_i)| - 2$

$W^{\Gamma}_{cu_2} \supseteq \{c\} \cup V(G_i) \Rightarrow 1 + |V(G_i)| \leq |W^{\Gamma}_{cu_2}|$

$W^{\Gamma}_{u_2c} \subseteq V(G_j) \setminus \{u_1\} \Rightarrow |W^{\Gamma}_{u_2c}| \leq |V(G_j)| - 1$

$\Rightarrow |V(G_i)| + 2 \leq |V(G_j)|$

So: $|V(G_i)| + 2 \leq |V(G_j)| \leq |V(G_i)| - 2$, contradiction.

From now on (w.l.o.g.): $c$ is adjacent to every vertex in $V(G_1)$.

# Proof - Claim 1

$$W^{\Gamma}_{cv_2} \supseteq \{c\} \cup V(G_j) \Rightarrow 1 + |V(G_j)| \leq |W^{\Gamma}_{cv_2}|$$

$$W^{\Gamma}_{v_2c} \subseteq V(G_i) \setminus \{v_1\} \Rightarrow |W^{\Gamma}_{v_2c}| \leq |V(G_i)| - 1$$

$$\Rightarrow |V(G_j)| \leq |V(G_i)| - 2$$

$$W^{\Gamma}_{cu_2} \supseteq \{c\} \cup V(G_i) \Rightarrow 1 + |V(G_i)| \leq |W^{\Gamma}_{cu_2}|$$

$$W^{\Gamma}_{u_2c} \subseteq V(G_j) \setminus \{u_1\} \Rightarrow |W^{\Gamma}_{u_2c}| \leq |V(G_j)| - 1$$

$$\Rightarrow |V(G_i)| + 2 \leq |V(G_j)|$$

So: $|V(G_i)| + 2 \leq |V(G_j)| \leq |V(G_i)| - 2$, contradiction.

From now on (w.l.o.g.): $c$ is adjacent to every vertex in $V(G_1)$.

# Proof - Claim 1

$$W^{\Gamma}_{cv_2} \supseteq \{c\} \cup V(G_j) \Rightarrow 1 + |V(G_j)| \leq |W^{\Gamma}_{cv_2}|$$

$$W^{\Gamma}_{v_2c} \subseteq V(G_i) \setminus \{v_1\} \Rightarrow |W^{\Gamma}_{v_2c}| \leq |V(G_i)| - 1$$

$$\Rightarrow |V(G_j)| \leq |V(G_i)| - 2$$

$$W^{\Gamma}_{cu_2} \supseteq \{c\} \cup V(G_i) \Rightarrow 1 + |V(G_i)| \leq |W^{\Gamma}_{cu_2}|$$

$$W^{\Gamma}_{u_2c} \subseteq V(G_j) \setminus \{u_1\} \Rightarrow |W^{\Gamma}_{u_2c}| \leq |V(G_j)| - 1$$

$$\Rightarrow |V(G_i)| + 2 \leq |V(G_j)|$$

So: $|V(G_i)| + 2 \leq |V(G_j)| \leq |V(G_i)| - 2$, contradiction.

From now on (w.l.o.g.): $c$ is adjacent to every vertex in $V(G_1)$.

# Proof - Claim 1

$$W^{\Gamma}_{cv_2} \supseteq \{c\} \cup V(G_j) \Rightarrow 1 + |V(G_j)| \leq |W^{\Gamma}_{cv_2}|$$

$$W^{\Gamma}_{v_2c} \subseteq V(G_i) \setminus \{v_1\} \Rightarrow |W^{\Gamma}_{v_2c}| \leq |V(G_i)| - 1$$

$$\Rightarrow |V(G_j)| \leq |V(G_i)| - 2$$

$$W^{\Gamma}_{cu_2} \supseteq \{c\} \cup V(G_i) \Rightarrow 1 + |V(G_i)| \leq |W^{\Gamma}_{cu_2}|$$

$$W^{\Gamma}_{u_2c} \subseteq V(G_j) \setminus \{u_1\} \Rightarrow |W^{\Gamma}_{u_2c}| \leq |V(G_j)| - 1$$

$$\Rightarrow |V(G_i)| + 2 \leq |V(G_j)|$$

So: $|V(G_i)| + 2 \leq |V(G_j)| \leq |V(G_i)| - 2$, contradiction.

From now on (w.l.o.g.): $c$ is adjacent to every vertex in $V(G_1)$.

# Proof - Claim 1

$W^{\Gamma}_{cv_2} \supseteq \{c\} \cup V(G_j) \Rightarrow 1 + |V(G_j)| \leq |W^{\Gamma}_{cv_2}|$

$W^{\Gamma}_{v_2c} \subseteq V(G_i) \setminus \{v_1\} \Rightarrow |W^{\Gamma}_{v_2c}| \leq |V(G_i)| - 1$

$\Rightarrow |V(G_j)| \leq |V(G_i)| - 2$

$W^{\Gamma}_{cu_2} \supseteq \{c\} \cup V(G_i) \Rightarrow 1 + |V(G_i)| \leq |W^{\Gamma}_{cu_2}|$

$W^{\Gamma}_{u_2c} \subseteq V(G_j) \setminus \{u_1\} \Rightarrow |W^{\Gamma}_{u_2c}| \leq |V(G_j)| - 1$

$\Rightarrow |V(G_i)| + 2 \leq |V(G_j)|$

So: $|V(G_i)| + 2 \leq |V(G_j)| \leq |V(G_i)| - 2$, contradiction.

From now on (w.l.o.g.): $c$ is adjacent to every vertex in $V(G_1)$.

# Proof - Claim 1

$W^{\Gamma}_{cv_2} \supseteq \{c\} \cup V(G_j) \Rightarrow 1 + |V(G_j)| \leq |W^{\Gamma}_{cv_2}|$

$W^{\Gamma}_{v_2c} \subseteq V(G_i) \setminus \{v_1\} \Rightarrow |W^{\Gamma}_{v_2c}| \leq |V(G_i)| - 1$

$\Rightarrow |V(G_j)| \leq |V(G_i)| - 2$

$W^{\Gamma}_{cu_2} \supseteq \{c\} \cup V(G_i) \Rightarrow 1 + |V(G_i)| \leq |W^{\Gamma}_{cu_2}|$

$W^{\Gamma}_{u_2c} \subseteq V(G_j) \setminus \{u_1\} \Rightarrow |W^{\Gamma}_{u_2c}| \leq |V(G_j)| - 1$

$\Rightarrow |V(G_i)| + 2 \leq |V(G_j)|$

So: $|V(G_i)| + 2 \leq |V(G_j)| \leq |V(G_i)| - 2$, contradiction.

From now on (w.l.o.g.): $c$ is adjacent to every vertex in $V(G_1)$.

# Proof - Claim 2

Claim 2: Induced subgraph $G_1$ is regular.

Observations: Pick arbitrary $u \in V(G) \setminus V(G_1)$ adjacent to $c$.

- $d(u, v) = 2$ for every $v \in V(G_1)$
- $|W^{\Gamma}_{ux}| = |W^{\Gamma}_{uy}|$ for arbitrary $x, y \in V(G_1)$
- $W^{\Gamma}_{vu} = \{v\} \cup (N_{\Gamma}(v) \setminus \{c\})$ for every $v \in V(G_1)$

  $\Rightarrow |W^{\Gamma}_{vu}| = 1 + |N_{\Gamma}(v)| - 1 = |N_{G_1}(v)| + 1$ for every $v \in V(G_1)$

$\Rightarrow |N_{G_1}(x)| + 1 = |N_{\Gamma}(x)| = |W^{\Gamma}_{xu}| = |W^{\Gamma}_{ux}| = |W^{\Gamma}_{uy}| = |W^{\Gamma}_{yu}| = |N_{\Gamma}(y)| = |N_{G_1}(y)| + 1$ for arbitrary $x, y \in V(G_1)$

From now on: Induced subgraph $G_1$ is regular with valency $k$. (Every vertex in $V(G_1)$ has valency $k + 1$ in $\Gamma$.)

# Proof - Claim 2

<u>Claim 2:</u> Induced subgraph $G_1$ is regular.

<u>Observations:</u> Pick arbitrary $u \in V(G) \setminus V(G_1)$ adjacent to $c$.

- $d(u, v) = 2$ for every $v \in V(G_1)$
- $|W^\Gamma_{ux}| = |W^\Gamma_{uy}|$ for arbitrary $x, y \in V(G_1)$
- $W^\Gamma_{vu} = \{v\} \cup (N_\Gamma(v) \setminus \{c\})$ for every $v \in V(G_1)$

  $\Rightarrow |W^\Gamma_{vu}| = 1 + |N_\Gamma(v)| - 1 = |N_{G_1}(v)| + 1$ for every $v \in V(G_1)$

$\Rightarrow |N_{G_1}(x)| + 1 = |N_\Gamma(x)| = |W^\Gamma_{xu}| = |W^\Gamma_{ux}| = |W^\Gamma_{uy}| = |W^\Gamma_{yu}| = |N_\Gamma(y)| = |N_{G_1}(y)| + 1$ for arbitrary $x, y \in V(G_1)$

<u>From now on:</u> Induced subgraph $G_1$ is regular with valency $k$. (Every vertex in $V(G_1)$ has valency $k + 1$ in $\Gamma$.)

# Proof - Claim 2

Claim 2: Induced subgraph $G_1$ is regular.

Observations: Pick arbitrary $u \in V(G) \setminus V(G_1)$ adjacent to $c$.

- $d(u, v) = 2$ for every $v \in V(G_1)$
- $|W^{\Gamma}_{ux}| = |W^{\Gamma}_{uy}|$ for arbitrary $x, y \in V(G_1)$
- $W^{\Gamma}_{vu} = \{v\} \cup (N_{\Gamma}(v) \setminus \{c\})$ for every $v \in V(G_1)$

  $\Rightarrow |W^{\Gamma}_{vu}| = 1 + |N_{\Gamma}(v)| - 1 = |N_{G_1}(v)| + 1$ for every $v \in V(G_1)$

$\Rightarrow |N_{G_1}(x)| + 1 = |N_{\Gamma}(x)| = |W^{\Gamma}_{xu}| = |W^{\Gamma}_{ux}| = |W^{\Gamma}_{uy}| = |W^{\Gamma}_{yu}| = |N_{\Gamma}(y)| = |N_{G_1}(y)| + 1$ for arbitrary $x, y \in V(G_1)$

From now on: Induced subgraph $G_1$ is regular with valency $k$.
(Every vertex in $V(G_1)$ has valency $k + 1$ in $\Gamma$.)

# Proof - Claim 2

Claim 2: Induced subgraph $G_1$ is regular.

Observations: Pick arbitrary $u \in V(G) \setminus V(G_1)$ adjacent to $c$.

- $d(u, v) = 2$ for every $v \in V(G_1)$
- $|W^\Gamma_{ux}| = |W^\Gamma_{uy}|$ for arbitrary $x, y \in V(G_1)$
- $W^\Gamma_{vu} = \{v\} \cup (N_\Gamma(v) \setminus \{c\})$ for every $v \in V(G_1)$

  $\Rightarrow |W^\Gamma_{vu}| = 1 + |N_\Gamma(v)| - 1 = |N_{G_1}(v)| + 1$ for every $v \in V(G_1)$

$\Rightarrow |N_{G_1}(x)| + 1 = |N_\Gamma(x)| = |W^\Gamma_{xu}| = |W^\Gamma_{ux}| = |W^\Gamma_{uy}| = |W^\Gamma_{yu}| = |N_\Gamma(y)| = |N_{G_1}(y)| + 1$ for arbitrary $x, y \in V(G_1)$

From now on: Induced subgraph $G_1$ is regular with valency $k$.
(Every vertex in $V(G_1)$ has valency $k + 1$ in $\Gamma$.)

# Proof - Claim 2

Claim 2: Induced subgraph $G_1$ is regular.

Observations: Pick arbitrary $u \in V(G) \setminus V(G_1)$ adjacent to $c$.

- $d(u, v) = 2$ for every $v \in V(G_1)$
- $|W^{\Gamma}_{ux}| = |W^{\Gamma}_{uy}|$ for arbitrary $x, y \in V(G_1)$
- $W^{\Gamma}_{vu} = \{v\} \cup (N_{\Gamma}(v) \setminus \{c\})$ for every $v \in V(G_1)$

  $\Rightarrow |W^{\Gamma}_{vu}| = 1 + |N_{\Gamma}(v)| - 1 = |N_{G_1}(v)| + 1$ for every $v \in V(G_1)$

$\Rightarrow |N_{G_1}(x)| + 1 = |N_{\Gamma}(x)| = |W^{\Gamma}_{xu}| = |W^{\Gamma}_{ux}| = |W^{\Gamma}_{uy}| = |W^{\Gamma}_{yu}| = |N_{\Gamma}(y)| = |N_{G_1}(y)| + 1$ for arbitrary $x, y \in V(G_1)$

From now on: Induced subgraph $G_1$ is regular with valency $k$.
(Every vertex in $V(G_1)$ has valency $k + 1$ in $\Gamma$.)

# Proof - Claim 2

Claim 2: Induced subgraph $G_1$ is regular.

Observations: Pick arbitrary $u \in V(G) \setminus V(G_1)$ adjacent to $c$.

- $d(u, v) = 2$ for every $v \in V(G_1)$
- $|W^{\Gamma}_{ux}| = |W^{\Gamma}_{uy}|$ for arbitrary $x, y \in V(G_1)$
- $W^{\Gamma}_{vu} = \{v\} \cup (N_{\Gamma}(v) \setminus \{c\})$ for every $v \in V(G_1)$

$\Rightarrow |W^{\Gamma}_{vu}| = 1 + |N_{\Gamma}(v)| - 1 = |N_{G_1}(v)| + 1$ for every $v \in V(G_1)$

$\Rightarrow |N_{G_1}(x)| + 1 = |N_{\Gamma}(x)| = |W^{\Gamma}_{xu}| = |W^{\Gamma}_{ux}| = |W^{\Gamma}_{uy}| = |W^{\Gamma}_{yu}| = |N_{\Gamma}(y)| = |N_{G_1}(y)| + 1$ for arbitrary $x, y \in V(G_1)$

From now on: Induced subgraph $G_1$ is regular with valency $k$. (Every vertex in $V(G_1)$ has valency $k + 1$ in $\Gamma$.)

# Proof - Claim 2

Claim 2: Induced subgraph $G_1$ is regular.

Observations: Pick arbitrary $u \in V(G) \setminus V(G_1)$ adjacent to $c$.

- $d(u, v) = 2$ for every $v \in V(G_1)$
- $|W_{ux}^{\Gamma}| = |W_{uy}^{\Gamma}|$ for arbitrary $x, y \in V(G_1)$
- $W_{vu}^{\Gamma} = \{v\} \cup (N_{\Gamma}(v) \setminus \{c\})$ for every $v \in V(G_1)$

  $\Rightarrow |W_{vu}^{\Gamma}| = 1 + |N_{\Gamma}(v)| - 1 = |N_{G_1}(v)| + 1$ for every $v \in V(G_1)$

$\Rightarrow |N_{G_1}(x)| + 1 = |N_{\Gamma}(x)| = |W_{xu}^{\Gamma}| = |W_{ux}^{\Gamma}| = |W_{uy}^{\Gamma}| = |W_{yu}^{\Gamma}| = |N_{\Gamma}(y)| = |N_{G_1}(y)| + 1$ for arbitrary $x, y \in V(G_1)$

From now on: Induced subgraph $G_1$ is regular with valency $k$.
(Every vertex in $V(G_1)$ has valency $k+1$ in $\Gamma$.)

# Proof - Claim 2

Claim 2: Induced subgraph $G_1$ is regular.

Observations: Pick arbitrary $u \in V(G) \setminus V(G_1)$ adjacent to $c$.

- $d(u, v) = 2$ for every $v \in V(G_1)$
- $|W_{ux}^{\Gamma}| = |W_{uy}^{\Gamma}|$ for arbitrary $x, y \in V(G_1)$
- $W_{vu}^{\Gamma} = \{v\} \cup (N_{\Gamma}(v) \setminus \{c\})$ for every $v \in V(G_1)$

  $\Rightarrow |W_{vu}^{\Gamma}| = 1 + |N_{\Gamma}(v)| - 1 = |N_{G_1}(v)| + 1$ for every $v \in V(G_1)$

$\Rightarrow |N_{G_1}(x)| + 1 = |N_{\Gamma}(x)| = |W_{xu}^{\Gamma}| = |W_{ux}^{\Gamma}| = |W_{uy}^{\Gamma}| = |W_{yu}^{\Gamma}| = |N_{\Gamma}(y)| = |N_{G_1}(y)| + 1$ for arbitrary $x, y \in V(G_1)$

From now on: Induced subgraph $G_1$ is regular with valency $k$. (Every vertex in $V(G_1)$ has valency $k+1$ in $\Gamma$.)

# Proof - Claim 3

Claim 3: $c$ is adjacent to every vertex in $V(G)$.

Suppose this statement is not true. Then:

$\exists u_2 \in V(G_2)$ s.t. $d(c, u_2) = 2$

$\Rightarrow \exists u_1 \in V(G_2)$ s.t. $d(c, u_1) = (d(u_1, u_2) = 1$

We know: $|N_\Gamma(v)| = k + 1$ for arbitrary $v$ in $V(G_1)$.

$W^\Gamma_{vu_1} = \{v\} \cup (N_\Gamma(v) \setminus \{c\})$

$|W^\Gamma_{vu_1}| = 1 + k + 1 - 1 = k + 1$

$W^\Gamma_{cu_2} \supseteq V(G_1) \cup \{c\}$

$|W^\Gamma_{cu_2}| \geq |V(G_1)| + 1 \geq k + 2$

# Proof - Claim 3

Claim 3: $c$ is adjacent to every vertex in $V(G)$.

Suppose this statement is not true. Then:

$\exists u_2 \in V(G_2)$ s.t. $d(c, u_2) = 2$

$\Rightarrow \exists u_1 \in V(G_2)$ s.t. $d(c, u_1) = (d(u_1, u_2) = 1$

We know: $|N_\Gamma(v)| = k + 1$ for arbitrary $v$ in $V(G_1)$.

$W^\Gamma_{vu_1} = \{v\} \cup (N_\Gamma(v) \setminus \{c\})$

$|W^\Gamma_{vu_1}| = 1 + k + 1 - 1 = k + 1$

$W^\Gamma_{cu_2} \supseteq V(G_1) \cup \{c\}$

$|W^\Gamma_{cu_2}| \geq |V(G_1)| + 1 \geq k + 2$

# Proof - Claim 3

Claim 3: $c$ is adjacent to every vertex in $V(G)$.

Suppose this statement is not true. Then:

$\exists u_2 \in V(G_2)$ s.t. $d(c, u_2) = 2$

$\Rightarrow \exists u_1 \in V(G_2)$ s.t. $d(c, u_1) = (d(u_1, u_2) = 1$

We know: $|N_\Gamma(v)| = k + 1$ for arbitrary $v$ in $V(G_1)$.

$W^\Gamma_{vu_1} = \{v\} \cup (N_\Gamma(v) \setminus \{c\})$

$|W^\Gamma_{vu_1}| = 1 + k + 1 - 1 = k + 1$

$W^\Gamma_{cu_2} \supseteq V(G_1) \cup \{c\}$

$|W^\Gamma_{cu_2}| \geq |V(G_1)| + 1 \geq k + 2$

# Proof - Claim 3

Claim 3: $c$ is adjacent to every vertex in $V(G)$.

Suppose this statement is not true. Then:

$\exists u_2 \in V(G_2)$ s.t. $d(c, u_2) = 2$

$\Rightarrow \exists u_1 \in V(G_2)$ s.t. $d(c, u_1) = (d(u_1, u_2) = 1$

We know: $|N_\Gamma(v)| = k + 1$ for arbitrary $v$ in $V(G_1)$.

$W^\Gamma_{vu_1} = \{v\} \cup (N_\Gamma(v) \setminus \{c\})$

$|W^\Gamma_{vu_1}| = 1 + k + 1 - 1 = k + 1$

$W^\Gamma_{cu_2} \supseteq V(G_1) \cup \{c\}$

$|W^\Gamma_{cu_2}| \geq |V(G_1)| + 1 \geq k + 2$

# Proof - Claim 3

Claim 3: $c$ is adjacent to every vertex in $V(G)$.

Suppose this statement is not true. Then:

$\exists u_2 \in V(G_2)$ s.t. $d(c, u_2) = 2$

$\Rightarrow \exists u_1 \in V(G_2)$ s.t. $d(c, u_1) = (d(u_1, u_2) = 1$

We know: $|N_\Gamma(v)| = k + 1$ for arbitrary $v$ in $V(G_1)$.

$W^\Gamma_{vu_1} = \{v\} \cup (N_\Gamma(v) \setminus \{c\})$

$|W^\Gamma_{vu_1}| = 1 + k + 1 - 1 = k + 1$

$W^\Gamma_{cu_2} \supseteq V(G_1) \cup \{c\}$

$|W^\Gamma_{cu_2}| \geq |V(G_1)| + 1 \geq k + 2$

# Proof - Claim 3

<u>Claim 3:</u> $c$ is adjacent to every vertex in $V(G)$.

Suppose this statement is not true. Then:

$\exists u_2 \in V(G_2)$ s.t. $d(c, u_2) = 2$

$\Rightarrow \exists u_1 \in V(G_2)$ s.t. $d(c, u_1) = (d(u_1, u_2) = 1$

<u>We know:</u> $|N_\Gamma(v)| = k + 1$ for arbitrary $v$ in $V(G_1)$.

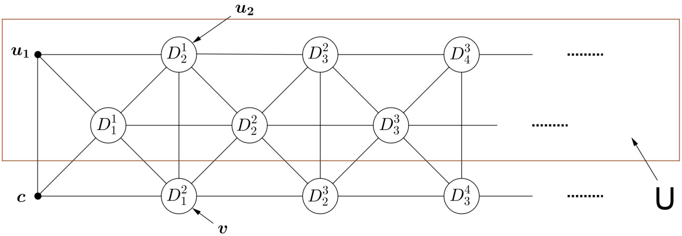
$W^\Gamma_{vu_1} = \{v\} \cup (N_\Gamma(v) \setminus \{c\})$

$|W^\Gamma_{vu_1}| = 1 + k + 1 - 1 = k + 1$

$W^\Gamma_{cu_2} \supseteq V(G_1) \cup \{c\}$

$|W^\Gamma_{cu_2}| \geq |V(G_1)| + 1 \geq k + 2$

# Proof - Claim 3



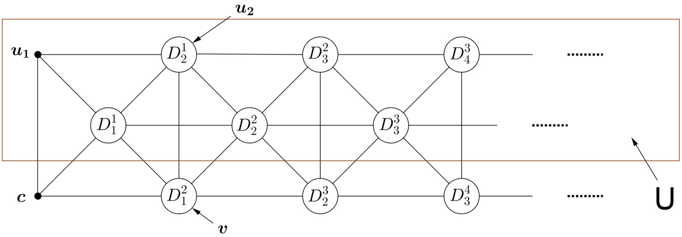

Define: $U = \bigcup_{i=1}^{d} = \left( D_i^{i-1} \cup D_i^i \right)$

$W_{u_2 c}^{\Gamma} \subseteq U \subseteq W_{u_1 v}^{\Gamma}$
$\Rightarrow |W_{u_2 c}^{\Gamma}| \leq |W_{u_1 v}^{\Gamma}| = k + 1$

$\Rightarrow k + 2 \leq |W_{c u_2}^{\Gamma}| = |W_{u_2 c}^{\Gamma}| \leq k + 1$, contradiction.

From now on: $c$ is adjacent to every vertex in $V(G)$.

# Proof - Claim 3




Define: $U = \bigcup_{i=1}^{d} = \left( D_i^{i-1} \cup D_i^i \right)$

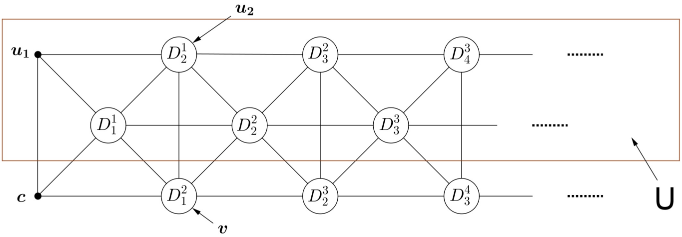
$W_{u_2 c}^{\Gamma} \subseteq U \subseteq W_{u_1 v}^{\Gamma}$

$\Rightarrow |W_{u_2 c}^{\Gamma}| \leq |W_{u_1 v}^{\Gamma}| = k+1$

$\Rightarrow k+2 \leq |W_{c u_2}^{\Gamma}| = |W_{u_2 c}^{\Gamma}| \leq k+1$, contradiction.

From now on: $c$ is adjacent to every vertex in $V(G)$.

# Proof - Claim 3




Define: $U = \bigcup_{i=1}^{d} = \left( D_i^{i-1} \cup D_i^i \right)$

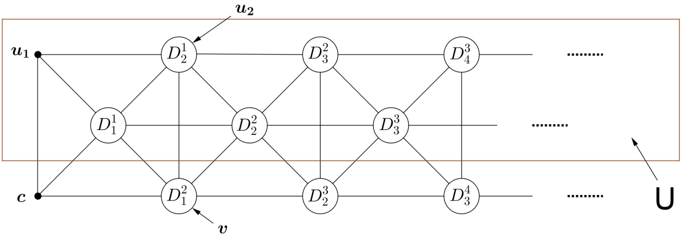
$W_{u_2 c}^{\Gamma} \subseteq U \subseteq W_{u_1 v}^{\Gamma}$

$\Rightarrow |W_{u_2 c}^{\Gamma}| \leq |W_{u_1 v}^{\Gamma}| = k + 1$

$\Rightarrow k + 2 \leq |W_{c u_2}^{\Gamma}| = |W_{u_2 c}^{\Gamma}| \leq k + 1$, contradiction.

From now on: $c$ is adjacent to every vertex in $V(G)$.

# Proof - Claim 3




Define: $U = \bigcup_{i=1}^{d} = \left(D_i^{i-1} \cup D_i^i\right)$

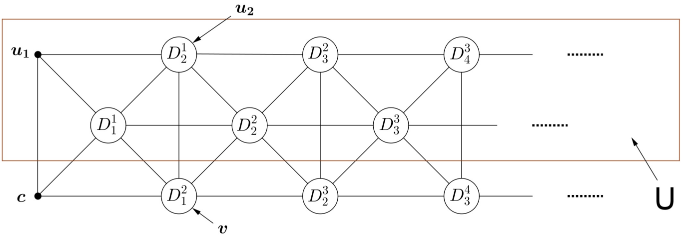
$W^{\Gamma}_{u_2 c} \subseteq U \subseteq W^{\Gamma}_{u_1 v}$

$\Rightarrow |W^{\Gamma}_{u_2 c}| \leq |W^{\Gamma}_{u_1 v}| = k + 1$

$\Rightarrow k + 2 \leq |W^{\Gamma}_{c u_2}| = |W^{\Gamma}_{u_2 c}| \leq k + 1$, contradiction.

From now on: $c$ is adjacent to every vertex in $V(G)$.

# Proof - Claim 3




Define: $U = \bigcup_{i=1}^{d} = \left(D_i^{i-1} \cup D_i^i\right)$

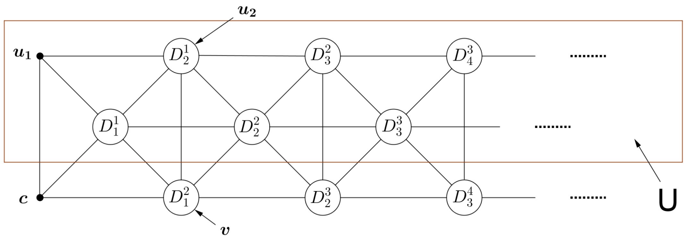
$W^{\Gamma}_{u_2 c} \subseteq U \subseteq W^{\Gamma}_{u_1 v}$

$\Rightarrow |W^{\Gamma}_{u_2 c}| \leq |W^{\Gamma}_{u_1 v}| = k + 1$

$\Rightarrow k + 2 \leq |W^{\Gamma}_{c u_2}| = |W^{\Gamma}_{u_2 c}| \leq k + 1$, contradiction.

From now on: $c$ is adjacent to every vertex in $V(G)$.

# Proof - Claim 3




Define: $U = \bigcup_{i=1}^{d} = \left( D_i^{i-1} \cup D_i^i \right)$

$W_{u_2 c}^{\Gamma} \subseteq U \subseteq W_{u_1 v}^{\Gamma}$

$\Rightarrow |W_{u_2 c}^{\Gamma}| \leq |W_{u_1 v}^{\Gamma}| = k + 1$

$\Rightarrow k + 2 \leq |W_{c u_2}^{\Gamma}| = |W_{u_2 c}^{\Gamma}| \leq k + 1$, contradiction.

From now on: $c$ is adjacent to every vertex in $V(G)$.

# Proof - Claim 4

<u>Claim 4:</u> Graph $G$ is regular with valency $k$.

W.l.o.g we prove that $G_2$ is regular with valency $k$.

Pick arbitrary $u \in V(G_2)$ and $v \in V(G_1)$ (we already now: $d(u, v) = 2$).

$W^{\Gamma}_{uv} = \{u\} \cup (N_{\Gamma}(u) \setminus \{c\}) = \{u\} \cup N_{G_2}(u)$

$W^{\Gamma}_{vu} = \{v\} \cup (N_{\Gamma}(v) \setminus \{c\}) = \{v\} \cup N_{G_1}(v)$

$\Rightarrow |W^{\Gamma}_{uv}| = 1 + |N_{G_2}(u)| \text{ in } |W^{\Gamma}_{vu}| = 1 + k.$

Since $|W^{\Gamma}_{uv}| = |W^{\Gamma}_{vu}|$

$\Rightarrow |N_{G_2}(u)| = k$ for every $u \in V(G_2)$

□

# Proof - Claim 4

Claim 4: Graph $G$ is regular with valency $k$.

W.l.o.g we prove that $G_2$ is regular with valency $k$.

Pick arbitrary $u \in V(G_2)$ and $v \in V(G_1)$ (we already now: $d(u, v) = 2$).

$W^{\Gamma}_{uv} = \{u\} \cup (N_{\Gamma}(u) \setminus \{c\}) = \{u\} \cup N_{G_2}(u)$

$W^{\Gamma}_{vu} = \{v\} \cup (N_{\Gamma}(v) \setminus \{c\}) = \{v\} \cup N_{G_1}(v)$

$\Rightarrow |W^{\Gamma}_{uv}| = 1 + |N_{G_2}(u)| \text{ in } |W^{\Gamma}_{vu}| = 1 + k.$

Since $|W^{\Gamma}_{uv}| = |W^{\Gamma}_{vu}|$

$\Rightarrow |N_{G_2}(u)| = k$ for every $u \in V(G_2)$

□

# Proof - Claim 4

Claim 4: Graph $G$ is regular with valency $k$.

W.l.o.g we prove that $G_2$ is regular with valency $k$.

Pick arbitrary $u \in V(G_2)$ and $v \in V(G_1)$ (we already now: $d(u, v) = 2$).

$W^{\Gamma}_{uv} = \{u\} \cup (N_{\Gamma}(u) \setminus \{c\}) = \{u\} \cup N_{G_2}(u)$

$W^{\Gamma}_{vu} = \{v\} \cup (N_{\Gamma}(v) \setminus \{c\}) = \{v\} \cup N_{G_1}(v)$

$\Rightarrow |W^{\Gamma}_{uv}| = 1 + |N_{G_2}(u)| \text{ in } |W^{\Gamma}_{vu}| = 1 + k.$

Since $|W^{\Gamma}_{uv}| = |W^{\Gamma}_{vu}|$

$\Rightarrow |N_{G_2}(u)| = k$ for every $u \in V(G_2)$

□

# Proof - Claim 4

Claim 4: Graph $G$ is regular with valency $k$.

W.l.o.g we prove that $G_2$ is regular with valency $k$.

Pick arbitrary $u \in V(G_2)$ and $v \in V(G_1)$ (we already now: $d(u,v) = 2$).

$W^{\Gamma}_{uv} = \{u\} \cup (N_{\Gamma}(u) \setminus \{c\}) = \{u\} \cup N_{G_2}(u)$

$W^{\Gamma}_{vu} = \{v\} \cup (N_{\Gamma}(v) \setminus \{c\}) = \{v\} \cup N_{G_1}(v)$

$\Rightarrow |W^{\Gamma}_{uv}| = 1 + |N_{G_2}(u)|$ in $|W^{\Gamma}_{vu}| = 1 + k.$

Since $|W^{\Gamma}_{uv}| = |W^{\Gamma}_{vu}|$

$\Rightarrow |N_{G_2}(u)| = k$ for every $u \in V(G_2)$

□

# Proof - Claim 4

Claim 4: Graph $G$ is regular with valency $k$.

W.l.o.g we prove that $G_2$ is regular with valency $k$.

Pick arbitrary $u \in V(G_2)$ and $v \in V(G_1)$ (we already now: $d(u, v) = 2$).

$$W^{\Gamma}_{uv} = \{u\} \cup (N_{\Gamma}(u) \setminus \{c\}) = \{u\} \cup N_{G_2}(u)$$

$$W^{\Gamma}_{vu} = \{v\} \cup (N_{\Gamma}(v) \setminus \{c\}) = \{v\} \cup N_{G_1}(v)$$

$$\Rightarrow |W^{\Gamma}_{uv}| = 1 + |N_{G_2}(u)| \quad \text{in} \quad |W^{\Gamma}_{vu}| = 1 + k.$$

Since $|W^{\Gamma}_{uv}| = |W^{\Gamma}_{vu}|$

$\Rightarrow |N_{G_2}(u)| = k$ for every $u \in V(G_2)$

$\square$

# Proof - Claim 4

Claim 4: Graph $G$ is regular with valency $k$.

W.l.o.g we prove that $G_2$ is regular with valency $k$.

Pick arbitrary $u \in V(G_2)$ and $v \in V(G_1)$ (we already now: $d(u,v) = 2$).

$$W_{uv}^{\Gamma} = \{u\} \cup (N_{\Gamma}(u) \setminus \{c\}) = \{u\} \cup N_{G_2}(u)$$

$$W_{vu}^{\Gamma} = \{v\} \cup (N_{\Gamma}(v) \setminus \{c\}) = \{v\} \cup N_{G_1}(v)$$

$$\Rightarrow |W_{uv}^{\Gamma}| = 1 + |N_{G_2}(u)| \text{ in } |W_{vu}^{\Gamma}| = 1 + k.$$

Since $|W_{uv}^{\Gamma}| = |W_{vu}^{\Gamma}|$

$\Rightarrow |N_{G_2}(u)| = k$ for every $u \in V(G_2)$

$\square$

# Proof - Claim 4

Claim 4: Graph $G$ is regular with valency $k$.

W.l.o.g we prove that $G_2$ is regular with valency $k$.

Pick arbitrary $u \in V(G_2)$ and $v \in V(G_1)$ (we already now: $d(u, v) = 2$).

$$W_{uv}^{\Gamma} = \{u\} \cup (N_\Gamma(u) \setminus \{c\}) = \{u\} \cup N_{G_2}(u)$$

$$W_{vu}^{\Gamma} = \{v\} \cup (N_\Gamma(v) \setminus \{c\}) = \{v\} \cup N_{G_1}(v)$$

$$\Rightarrow |W_{uv}^{\Gamma}| = 1 + |N_{G_2}(u)| \ \text{ in } \ |W_{vu}^{\Gamma}| = 1 + k.$$

Since $|W_{uv}^{\Gamma}| = |W_{vu}^{\Gamma}|$

$\Rightarrow |N_{G_2}(u)| = k$ for every $u \in V(G_2)$

$\square$

# 2-distance-balanced cartesian product $G \square H$

Question:

Which cartesian products $G \square H$ are 2-distance-balanced?

Cartesian product of graphs $G$ and $H$, denoted by $G \square H$, is a graph with vertex set $V(G) \times V(H)$, where $(u_1, v_1)$ and $(u_2, v_2)$ are adjacent if and only if either

- $u_1 = u_2$ and $v_1$ is adjacent to $v_2$ in $H$, or
- $v_1 = v_2$ and $u_1$ is adjacent to $u_2$ in $G$.

# 2-distance-balanced cartesian product $G \Box H$

Question:

Which cartesian products $G \Box H$ are 2-distance-balanced?

Cartesian product of graphs $G$ and $H$, denoted by $G \Box H$, is a graph with vertex set $V(G) \times V(H)$, where $(u_1, v_1)$ and $(u_2, v_2)$ are adjacent if and only if either

- $u_1 = u_2$ and $v_1$ is adjacent to $v_2$ in $H$, or
- $v_1 = v_2$ and $u_1$ is adjacent to $u_2$ in $G$.

# 2-distance-balanced cartesian product $G \Box H$

**Theorem (B.F., Š. Miklavič)**

Cartesian product $G \Box H$ is 2-distance-balanced iff one of the following statements is true:

(i) Both graphs $G$ an $H$ are 2-distance-balanced and 1-distance-balanced.

(ii) $G$ is a complete graph $K_n$ for some $n \geq 2$ and $H$ is a connected 2-distance-balanced and 1-distance-balanced graph.

(iii) $H$ is a complete graph $K_n$ for some $n \geq 2$ and $G$ is a connected 2-distance-balanced and 1-distance-balanced graph.

(iv) $G$ is a complete graph $K_n$ and $H$ is a complete graph $K_m$ for some $m, n \geq 2$.

# 2-distance-balanced lexicographic product $G[H]$

Question:

Which lexicographic products $G[H]$ are 2-distance-balanced?

Lexicographic product of graphs $G$ and $H$, denoted by $G[H]$, is a graph with vertex set $V(G) \times V(H)$, where $(u_1, v_1)$ and $(u_2, v_2)$ are adjacent if and only if either

- $u_1$ is adjacent to $u_2$ in $G$, or
- $u_1 = u_2$ and $v_1$ is adjacent to $v_2$ in $H$.

# 2-distance-balanced lexicographic product $G[H]$

Question:

Which lexicographic products $G[H]$ are 2-distance-balanced?

Lexicographic product of graphs $G$ and $H$, denoted by $G[H]$, is a graph with vertex set $V(G) \times V(H)$, where $(u_1, v_1)$ and $(u_2, v_2)$ are adjacent if and only if either

- $u_1$ is adjacent to $u_2$ in $G$, or
- $u_1 = u_2$ and $v_1$ is adjacent to $v_2$ in $H$.

# 2-distance-balanced lexicographic product $G[H]$

**Theorem (B.F., Š. Miklavič)**

Lexicographic product $G[H]$ is 2-distance-balanced iff one of the following statements is true:

(i) $G$ is a connected 2-distance-balanced graph and $H$ is a regular graph.

(ii) $G$ is a complete graph and $H$ is a regular graph, which is not complete.

(iii) $G$ is a complete graph and $H$ is a connected complete bipartite graph.

Thank you!!!