On bipartite Q-polynomial distance-regular graphs with $c_2 \leq 2$

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2015 International conference on Graph Theory
Koper, May 26-28, 2015

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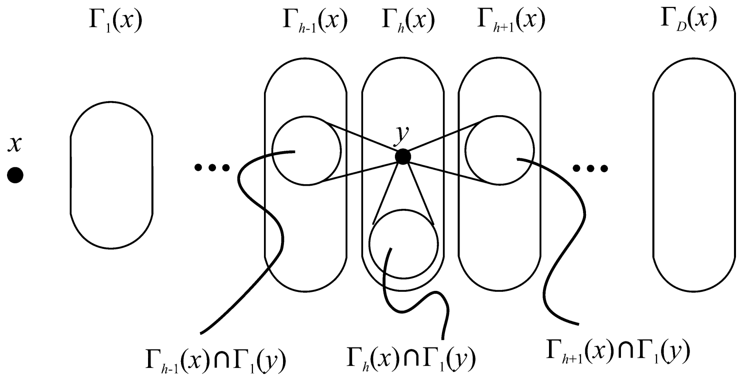
Outline

Some notation before definition of DRG



Distance-regular graphs

- A connected graph Γ is called distance-regular (DRG) if there are numbers a_i, b_i, c_i $(0 \leq i \leq D)$ s.t. if $\partial(x, y) = h$ then
  - $|\Gamma_1(y) \cap \Gamma_{h-1}(x)| = c_h$
  - $|\Gamma_1(y) \cap \Gamma_h(x)| = a_h$
  - $|\Gamma_1(y) \cap \Gamma_{h+1}(x)| = b_h$
- Numbers a_i, b_i and c_i $(0 \leq i \leq D)$ are called intersection numbers, and $\{b_0, b_1, ..., b_{D-1}; c_1, c_2, ..., c_D\}$ is intersection array.

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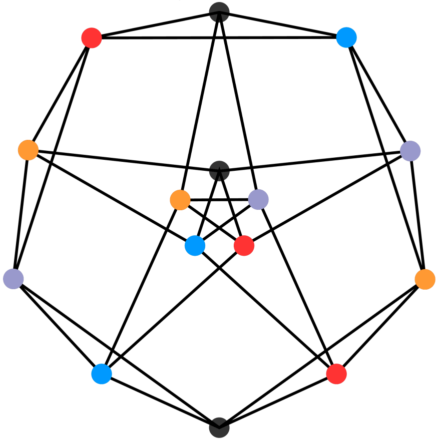
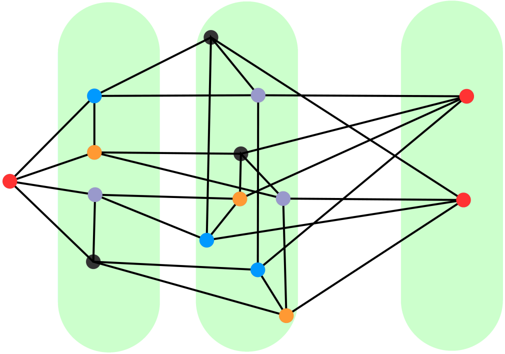
Distance-regular graphs - examples

- Line graph of Petersen's graph.



Distance-regular graphs - examples

- Line graph of Petersen's graph (diameter is three and intersection array is $\{4, 2, 1; 1, 1, 4\}$)



Hamming graphs

- The Hamming graph $H(n, q)$ is the graph whose vertices are words (sequences or n-tuples) of length n from an alphabet of size $q \geq 2$. Two vertices are considered adjacent if the words (or n-tuples) differ in exactly one term. We observe that $|V(H(n, q))| = q^n$.
- The Hamming graph $H(n, q)$ is distance-regular (with $a_i = i(q-2)$ $(0 \leq i \leq n)$, $b_i = (n-i)(q-1)$ $(0 \leq i \leq n-1)$ and $c_i = i$ $(1 \leq i \leq n)$).

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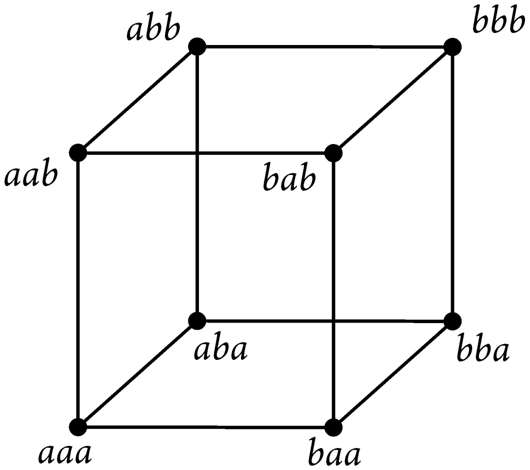

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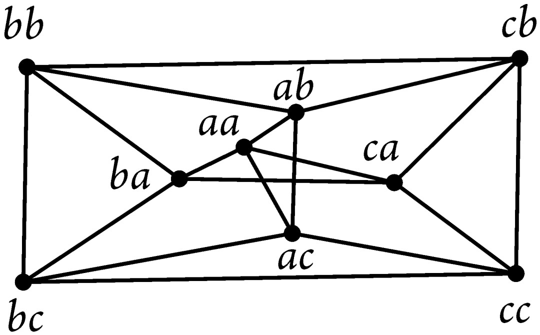
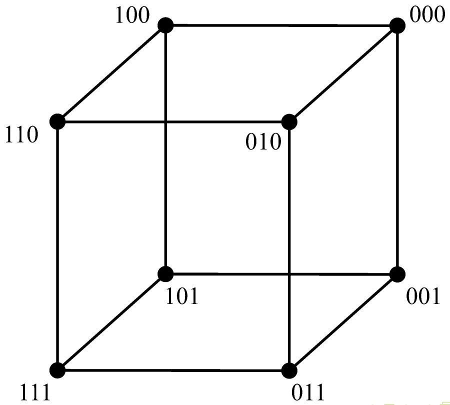
Hamming graphs $H(2, 3)$



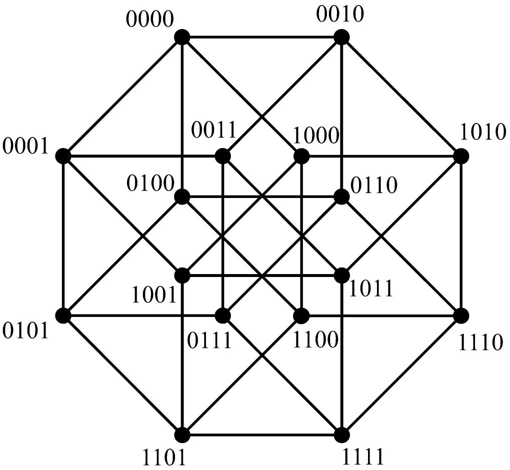
- Hamming graph $H(2, 3)$.

n-dimensional hypercubes (shortly n-cubes)

- Hamming graph $H(n, q)$ in which words of length n are from an alphabet of size $q = 2$ are called n-dimensional hypercubes or shortly n-cubes.



4-dimensional hypercube (4-cubes)



- 4-dimensional hypercube

More examples

- That comes from classical objects:
  - Hamming graphs,
  - Johnson graphs,
  - Grassmann graphs,
  - bilinear forms graphs,
  - sesquilinear forms graphs,
  - dual polar graphs (the vertices are the maximal totally isotropic subspaces on a vector space over a finite field with a fixed (non-degenerate) bilinear form)
- Some non-classical examples:
  - Doob graphs,
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- Distance-regular graphs give a way to study these classical objects from a combinatorial view point.

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Distance-i matrix

- Let $\mathrm{Mat}_V(\mathbb{R})$ denote the algebra of matrices over $\mathbb{R}$ with rows and columns indexed by V.
- For $0 \leq i \leq D$, let $\boldsymbol{A}_i$ denote the matrix in $\mathrm{Mat}_V(\mathbb{R})$ with (y, z)-entry

$$(\boldsymbol{A}_i)_{yz} = \begin{cases} 1 & \text{if } \partial(y,z) = i, \\ 0 & \text{if } \partial(y,z) \neq i \end{cases} \qquad (y, z \in X).$$

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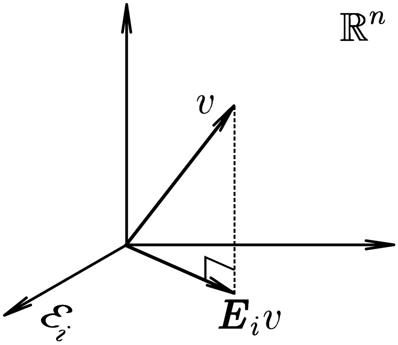
Primitive idempotents

- We refer to $\boldsymbol{E}_0$, ..., $\boldsymbol{E}_D$ as the primitive idempotents of Γ.
- Primitive idempotents of Γ represents the orthogonal projectors onto $\mathcal{E}_i = \ker(\boldsymbol{A} - \theta_i \boldsymbol{I})$ (along $\operatorname{im}(\boldsymbol{A} - \theta_i \boldsymbol{I})$)



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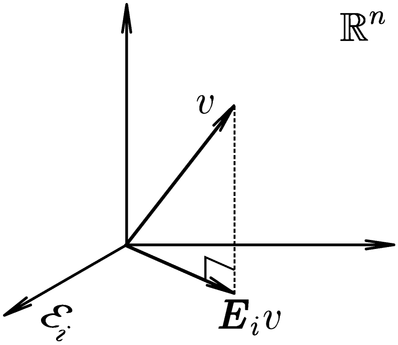
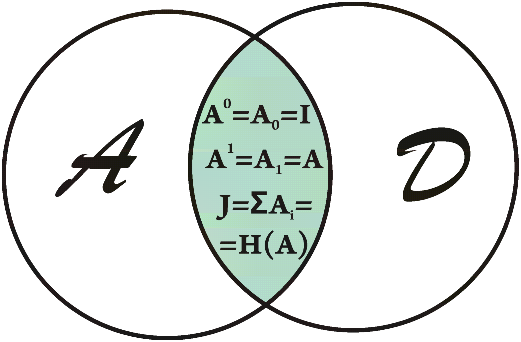


Distance algebra

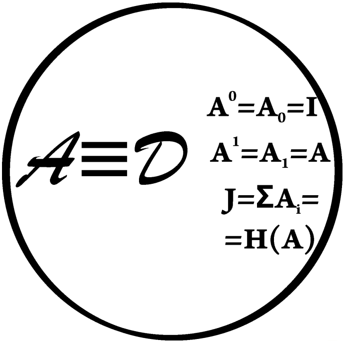
- If Γ is regular (and Γ is not distance-regular) we have:



- Adjacency algebra (ordinary "·" product), $\mathcal{A} = \mathrm{span}\{\boldsymbol{A}^0, \boldsymbol{A}^1, \ldots, \boldsymbol{A}^d\} = \mathrm{span}\{\boldsymbol{E}_0, \boldsymbol{E}_1, \ldots, \boldsymbol{E}_d\}$
- Distance algebra (entry-wise "∘" multiplication), $\mathcal{D} = \mathrm{span}\{\boldsymbol{A}_0, \boldsymbol{A}_1, \ldots, \boldsymbol{A}_D\}$

Case wehen Γ is is distance-regular

- The following statements are equivalent:
  (i) Γ is distance-regular,
  (ii) $\mathcal{D}$ is an algebra with the ordinary product,
  (iii) $\mathcal{A}$ is an algebra with the Hadamard product,
  (iv) $\mathcal{A} = \mathcal{D}$.



Q-polynomial property

- Let Γ denote any distance regular graph with diameter $D \geq 3$, and let $\boldsymbol{A}$ denote the adjacency algebra for Γ. Let $\boldsymbol{E}$ denote a primitive idempotent of Γ.
- Since $\mathcal{A}$ has a basis $\boldsymbol{A}_0$, $\boldsymbol{A}_1$, ..., $\boldsymbol{A}_D$ of $0-1$ matrices, $\mathcal{A}$ is closed under entry-wise matrix multiplication.
- Γ is said to be Q-polynomial with respect to $\boldsymbol{E} = \boldsymbol{E}_1$ whenever there exist an ordering $\boldsymbol{E}_0$, $\boldsymbol{E}_1$, ..., $\boldsymbol{E}_D$ of the primitive idempotents such that for each i $(0 \leq i \leq D)$, the primitive idempotent $\boldsymbol{E}_i$ is a polynomial of degree exactly i in $\boldsymbol{E}_1$, in the $\mathbb{R}$-algebra $(\mathcal{A}, \circ)$, where $\circ$ denote entry-wise multiplication.
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Result of Coughman, motivation

Theorem (Caughman, 2004)

Let Γ denote a bipartite distance-regular graph with diameter $D \geq 12$. If Γ is Q-polynomial then Γ is either the ordinary $2D$-cycle, or the D-dimensional hypercube, or the antipodal quotient of the $2D$-dimensional hypercube, or the intersection numbers of Γ satisfy $c_i = (q^i - 1)/(q - 1)$, $b_i = (q^D - q^i)/(q - 1)$ $(0 \leq i \leq D)$ for some integer q at least 2.

- Note that if $c_2 \leq 2$, then the last of the above possibilities cannot occur.
- It is the aim of this presentation to further investigate graphs with $D \leq 11$ and $c_2 \leq 2$.

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Result of Coughman, motivation (cont.)

- Our main result is the following theorem.

Theorem 1.

Let Γ denote a bipartite Q-polynomial distance-regular graph with diameter $D \geq 4$, valency $k \geq 3$, and intersection number $c_2 \leq 2$. Then one of the following holds:

(i) Γ is the D-dimensional hypercube;

(ii) Γ is the antipodal quotient of the $2D$-dimensional hypercube;

(iii) Γ is a graph with $D = 5$ not listed above.

Theorem 7.

- Let Γ denote a Q-polynomial bipartite distance-regular graph with diameter $D \geq 6$, valency $k \geq 3$, and intersection numbers b_i, c_i.
- In this section we show that if $c_2 \leq 2$, then Γ is either the D-dimensional hypercube, or the antipodal quotient of the $2D$-dimensional hypercube.

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Idea for proof of Theorem 7.

- Assume that Γ is not the D-dimensional hypercube or the antipodal quotient of the $2D$-dimensional hypercube.
- Then there exist scalars $s^*, q \in \mathbb{R}$ such that

$$c_i = \frac{h(q^i - 1)(1 - s^* q^{D+i+1})}{1 - s^* q^{2i+1}}, \qquad b_i = \frac{h(q^D - q^i)(1 - s^* q^{i+1})}{1 - s^* q^{2i+1}}$$

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Idea for proof of Theorem 7. (cont.)

- By [3, Lemma 4.1 and Lemma 5.1], scalars s^* and q satisfy

$$q > 1, \quad \text{and} \quad -q^{-D-1} \leq s^* < q^{-2D-1}. \tag{1}$$

- Assume first $c_2 = 1$. Abbreviate $\alpha = 1 + q - q^2 - q^{D-1} + q^D + q^{D+1}$ and observe $\alpha > 2$. By Lemma 6(iii) we find

$$s^* = \frac{\alpha \pm \sqrt{\alpha^2 - 4q^{D+1}}}{2q^{D+3}}.$$

- Note that $\alpha^2 - 4q^{D+1} \geq 0$, and so we have

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Definition of D^i_j

- Assume that $\Gamma = (X, R)$ is bipartite with diameter $D \geq 4$, valency $k \geq 3$ and intersection number $c_2 = 2$.
- In this section we describe certain partition of the vertex set X.

Definition 8.

Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$, valency $k \geq 3$ and intersection number $c_2 = 2$. Fix vertices $x, y \in X$ such that $\partial(x, y) = 2$. For all integers i, j we define $D^i_j = D^i_j(x, y)$ by

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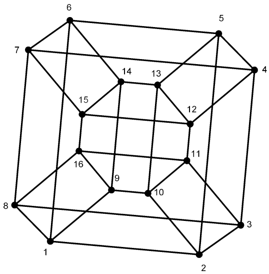
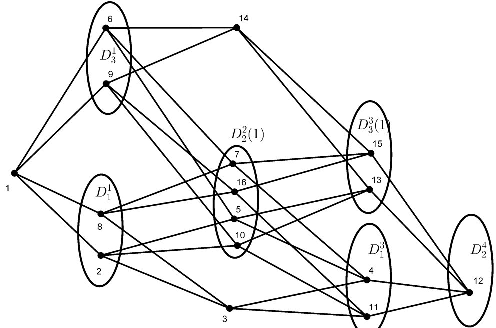
We observe $D^i_j = \emptyset$ unless $0 \leq i, j \leq D$. Moreover $|D^i_j| = p^2_{ij}$ for $0 \leq i, j \leq D$.

Definition of D_j^i - examples

- 4-cube with sets D_j^i ($b_0 = 4,\ b_1 = 3,\ b_2 = 2,\ b_3 = 1;\ c_1 = 1,\ c_2 = 2,\ c_3 = 3,\ c_4 = 4$).





Case $c_2 = 2$

- What if $c_2 = 2$?

Definition 13.

... For $1 \leq i \leq D$ we define $\mathcal{A}_i = \mathcal{A}_i(x,y)$, $\mathcal{C}_i = \mathcal{C}_i(x,y)$, $\mathcal{B}_i(z) = \mathcal{B}_i(z)(x,y)$, $\mathcal{B}_i(v) = \mathcal{B}_i(v)(x,y)$ by

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We observe $\mathcal{D}_i^i$ is a disjoint union of $\mathcal{A}_i, \mathcal{B}_i(z), \mathcal{B}_i(v), \mathcal{C}_i$.

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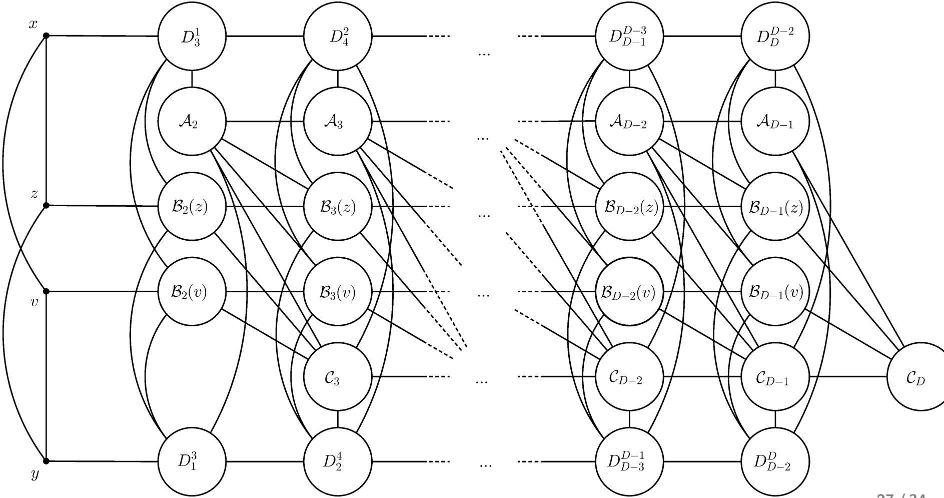
Case $c_2 = 2$ (cont.)

- Partition of graph Γ, which involves $4(D-1) + 2\ell$ cells



Equitable partition

- We claim that the partition of $V\Gamma$ into nonempty sets $D_{i+1}^{i-1}, D_{i-1}^{i+1}$ $(1 \leq i \leq D-1)$, $\mathcal{A}_i$ $(2 \leq i \leq D-1)$, $\mathcal{B}_i(z), \mathcal{B}_i(v)$ $(1 \leq i \leq D-1)$ and $\mathcal{C}_i$ $(3 \leq i \leq D)$ is equitable.
- Main tool is "balanced set theorem".

Theorem (Terwilliger, 1995) (abridged version of theorem)

Let Γ denote a distance-regular graph with diameter $D \geq 3$. Let E denote a nontrivial primitive idempotent of Γ and let $\{\theta_i^*\}_{i=0}^{D}$ denote the corresponding dual eigenvalue sequence.... Then for all integers h, i, j $(1 \leq h \leq D)$, $(0 \leq i, j \leq D)$ and for all $x, y \in X$ such that $\partial(x, y) = h$,

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Case $D = 4$

- In this section we consider Q-polynomial bipartite distance-regular graph Γ with intersection number $c_2 \leq 2$, valency $k \geq 3$ and diameter $D = 4$.
- We show that Γ is either the 4-dimensional hypercube, or the antipodal quotient of the 8-dimensional hypercube.

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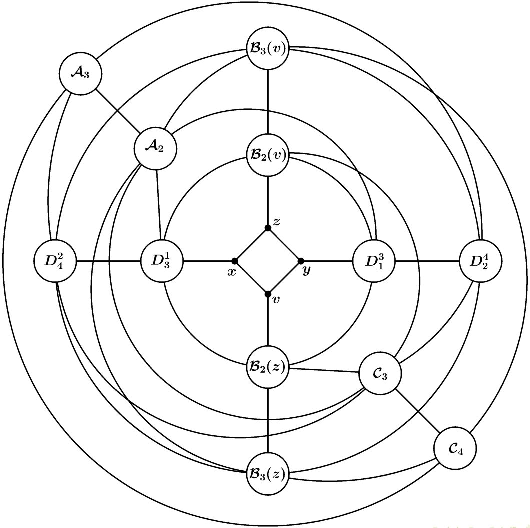
$c_2 = 2$ - Equitable partition



$c_2 = 2$ - ingredients

- Let Γ denote a Q-polynomial bipartite distance-regular graph with diameter $D = 4$, valency $k \geq 4$ and intersection number $c_2 = 2$. Assume Γ is not the 4-dimensional hypercube or the antipodal quotient of the 8-dimensional hypercube.
- $|\mathcal{A}_2| = (k-2)(c_3-3)/2$;
- $c_3 \geq 4$ if and only if $\mathcal{A}_2 \neq \emptyset$;
- pick $w \in \mathcal{A}_2$ let λ denote number or neighbours of w in $\mathcal{A}_3$;
- $\lambda = \dfrac{(k-2)b_3(b_3-1)}{(k-2)(k-3)-2b_3}$;
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- Each vertex in $\mathcal{B}_3(v)$ has exactly $\dfrac{(c_3-3)(b_3-\lambda)}{b_3}$ neighbours in $\mathcal{A}_2$.
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- $(k-2)(k-3)=2b_3^2$;
- $\lambda=(k-2)/2$;
- $q=-(\sqrt{5}+3)/2$;
- $s^*=72\sqrt{5}-161$.

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Theorem 35.

Let Γ denote a Q-polynomial bipartite distance-regular graph with diameter $D = 4$, valency $k \geq 3$ and intersection number $c_2 = 2$. Then Γ is either the 4-dimensional hypercube, or the antipodal quotient of the 8-dimensional hypercube.

- Assume first that $c_3 \geq 4$. Then by Lemma 34 we have $q = -(\sqrt{5} + 3)/2$ and $s^* = 72\sqrt{5} - 161$. Lemma 6(iii) now implies $k = -6$, a contradiction. Therefore $c_3 = 3$. But now [4, Theorem 4.6] implies that Γ is either the 4-dimensional hypercube, or the antipodal quotient of the 8-dimensional hypercube.

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[1] E. Bannai, T. Ito, *Algebraic Combinatorics I: Association Schemes*, The Benjamin-Cummings Lecture Notes Ser. 58, Menlo Park, CA, 1984.

[2] A. E. Brouwer, A. M. Cohen, and A. Neumaier, *Distance-Regular Graphs*, Springer-Verlag, Berlin, Heidelberg, 1989.

[3] J. S. Caughman, Bipartite Q-polynomial distance-regular graphs, *Graphs Combin.*, **20** (2004), 47-57.

[4] B. Curtin, Almost 2-homogeneous Bipartite Distance-regular Graphs, *European J. Combin.* **21** (2000), 865–876.

[5] E. R. van Dam, J. H. Koolen, and H. Tanaka, *Distance-regular graphs*, preprint, arXiv:1410.6294.

[6] Š. Miklavič, On bipartite Q-polynomial distance-regular graphs with $c_2 = 1$, *Discrete Math.*, **307** (2007), 544-553.

[7] P. Terwilliger, A new inequality for distance-regular graphs, *Discrete Math.*, **137** (1995), 319-332.