Some applications of generalized action graphs and monodromy graphs

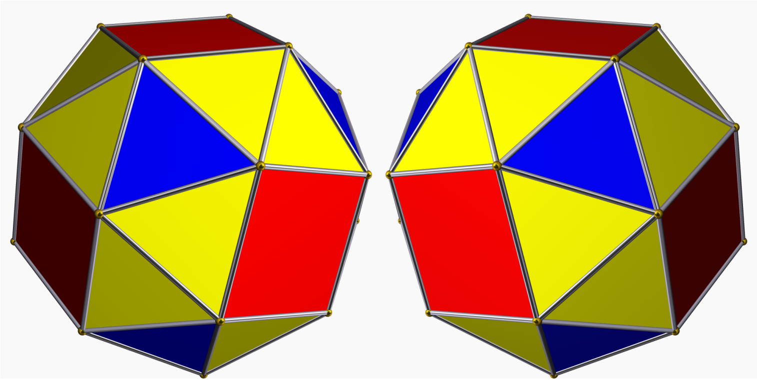
Tomaž Pisanski
(University of Primorska and University of Ljubljana)

2015 International Conference on Graph Theory
UP FAMNIT, Koper, May 26–28, 2015

Thanks

Thanks to my former students Alen Orbanić, María del Río-Francos and their and my co-authors, in particular to Jan Karabáš and many others whose work I am freely using in this talk.

Snub Cube - Chirality in the sense of Conway

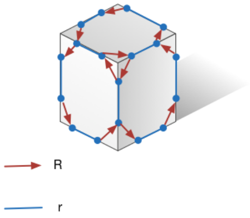


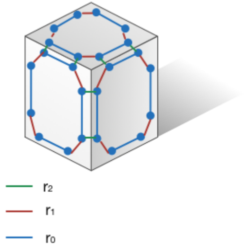
Snube cube is vertex-transitive (uniform) chiral polyhedron. It comes in two oriented forms.

Motivation

We would like to have a single combinatorial mechanism that would enable us to deal with such phenomena. Our goal is to unify several existing structures in such a way that the power of each individual structure is preserved.

Example: The Cube





Example

The cube may be considered as

- an *oriented map*: R, r,
- as a *map*: r_0, r_1, r_2
- as a polyhedron
- ....

Question

What is the dual of a cube and how can we describe it?

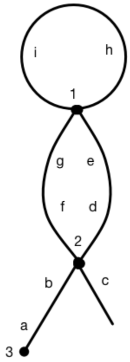
Question

How can we tell that the cube is a regular map and amphichiral oriented map?

Plan

We define *generalized action graphs* as semi-directed graphs in which the edge set is partitioned into directed 2-factors (forming an action digraph) and undirected 1-factors (forming a monodromy graph) and use them to describe several combinatorial structures, such as maps and oriented maps. The quotient of the action graph with respect to its automorphism group (or some of its subgroup) is called the *symmetry type graph* and is very useful in connection with map symmetries and orientation preserving symmetries. Several usual cases of regular, edge-transitive, vertex-transitive, chiral, etc. maps and oriented maps are revisited. Our symmetry type graphs are closely related to Delaney-Dress symbols and orbifolds. The theory is very general and applies to a variety of discrete structures such as hypermaps, abstract polytopes and maniplexes.

Pregraphs and Graphs

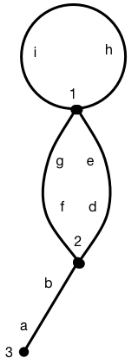


	a	b	c	d	e	f	g	h	i
r	b	a	c	e	d	g	f	i	h
i	3	2	2	2	1	2	1	1	1

A *pre-graph* Γ is a quadruple (V, D, i, r) where V and D are disjoint non-empty sets

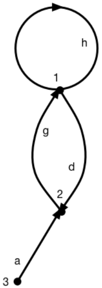
- V being the set of vertices,
- D the set of darts,
- $i : D \to V$ is a mapping that assigns to each dart its initial vertex and
- $r : D \to D$ is an involution assigning each dart its reverse dart.

Pregraphs and Graphs



	a	b	d	e	f	g	h	i
r	b	a	e	d	g	f	i	h
i	3	2	2	1	2	1	1	1

Let $\Gamma = (V, D, i, r)$ be a pre-graph. The orbits of r are called *edges* of Γ. The edges corresponding to fixed points of r are called *pending edges* or *semi-edges*, while other edges are called *proper edges*.
A pre-graph $\Gamma = (V, D, i, r)$ is *graph*, if r is fixed-point free.

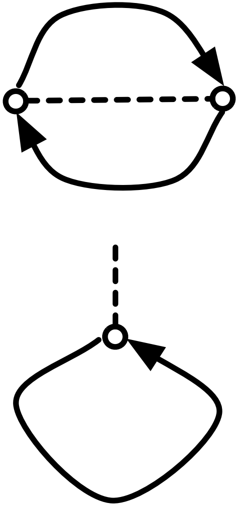


	a	d	g	h
j	3	1	2	1
t	2	2	1	1

A *digraph* Γ is a quadruple (V, A, j, t) where V and A are disjoint non-empty sets

- V being the set of vertices,
- A the set of arcs,
- $j : A \to V$ is a mapping that assigns to each dart its initial vertex and
- $t : A \to V$ is a mapping that assigns to each dart its terminal vertex.

Partially Directed Graphs



A *partially directed graph* Γ is a hextuple (V, D, A, i, r, j, t) where (V, D, i, r) is a pre-graph and (V, A, j, t) is a digraph.

Morphisms, Isomorphisms, Automorphisms

A morphism ϑ between two partially directed graphs (V, D, A, i, r, j, t) and $(V', D', A', i', r', j', t')$ satisfies the following, for any $d \in D$ and $a \in A$:

$$i'(\vartheta(d)) = \vartheta(i(d))$$

$$r'(\vartheta(d)) = \vartheta(r(d))$$

$$j'(\vartheta(a)) = \vartheta(j(a))$$

$$t'(\vartheta(a)) = \vartheta(t(a))$$

Action graphs

We generalize and adapt the notion of *action graph*, first introduced by A. Malnič in 2002 as follows.

- Let Φ be a finite non-empty set of *flags* or *vertices*.
- Let $\mathcal{R} = [R_1, R_2, \ldots, R_m]$, where $m \geq 0$ be a collection of permutations $\forall i \in I \colon R_i \in \mathrm{Sym}(V),\ I = \{1, 2, \ldots, m\}$, called (the set of) *rotations*.
- Let $\varrho = [r_1, r_2, \ldots, r_n]$ be a collection of involutions on Φ, i.e. $\forall j \in J \colon r_j \in \mathrm{Sym}(V), r_j^2 = \mathrm{Id},\ J = \{1, 2, \ldots, m\}$, called *reflections*.

The structure $\Gamma = (\Phi; \mathcal{R}; \varrho, I, J)$ is called an *action graph* on the vertex set Φ of *type* (I, J) with *signature* $(m, n), m = |I|, n = |J|$.

Action graphs

In practice we omit the type or replace it by signature when it is not needed.

The subgroup $\mathrm{Mon}\,\Gamma = \langle \mathcal{R}; \varrho \rangle \leq \mathrm{Sym}(\Phi)$ is called the *monodromy group* of Γ.

The elements of $\mathrm{Mon}\,\Gamma$ act from right on the elements of Φ, as it is defined by the convention.

The action graph Γ is *connected* if $\mathrm{Mon}\,\Gamma$ acts transitively on Φ.

Action graphs

The action graph $\Gamma = (\Phi; \mathcal{R}, \rho)$ can be viewed as a partially directed regular graph on the vertex set Φ with edges coloured by the corresponding generators of $\mathrm{Mon}\,\Gamma$. The incidence relation in Γ reads as $u \sim v \iff v = u.g$, where g is a generator of $\mathrm{Mon}\,\Gamma$. The valence of Γ with signature (m, n) is $2m + n$. A fixed point of R_i is exhibited as a *loop*, while a fixed point of r_j is considered as a *semiedge*.

We will always represent a pair of the opposite arcs arising from the action of reflections by the undirected edge; hence Γ is partially directed.

Example - Cayley graphs as action graphs

Example

A (directed) Cayley graph with $m+n$ generators, out of which there are n involutions, can be regarded as an action graph with signature $(m+k, n-k), 0 \leq k \leq n$. Note that an involution may play different roles. It may be counted as permutation or as involution. This explains different signatures.

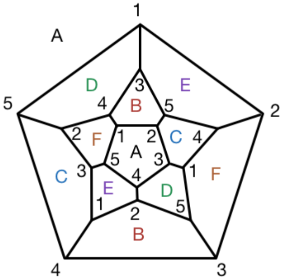
Example - Abstract polytopes as action graphs

Example

An abstract polytope of rank n is an action graph with signature $(0, n)$. The vertices are flags and the n involutions represent exchange maps on the flags of the polytope:
$\Gamma = (\Phi; \emptyset, \{r_0, r_1, \ldots, r_{n-1}\}), r_i r_j = r_j r_i, |i - j| > 2$. For Γ to represent an abstract polytopes two further technical conditions have to be met: the *diamond condition* and *strong connectivity*.

Instead of abstract polytopes one could study more general *maniplexes* (without diamond condition and strong connectivity). Maniplexes were introduced by Steve Wilson in 2012. See also *crystallizations* by Ferri and Gagliardi, introduced in the 70s, not to forget Lins and his work in the 90s.

Maniplex vs. Abstract Polytope



Dodecahedral Space

$$v = 5, e = 10, f = 6, F = 1$$

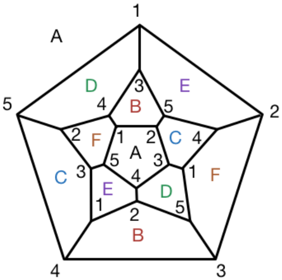
There are 120 flags! Triple (vertex-edge-face) gives rise to two flags!

Proposition

Every abstract polytope is a maniplex.

Proposition

There exist a maniplex that is not an abstract polytope, e.g. the Poincaré Homology Sphere,



Dodecahedral Space alias Poincaré Homology 3-Sphere.

The 120 flags are vertices of the dual of the barycentric subdivision of the ordinary dodecahedron on the sphere. The flag graph of the dodecahedron is a trivalent 0-,1-,2- edge-colored spanning subgraph of the flag graph of the dodecahedral space. 3-edges connect corresponding flags in antipodal pentagonal faces.

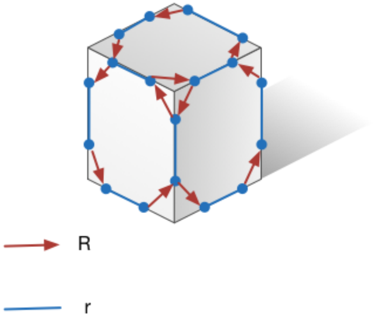
Question about maniplexes.

Question

Is there an easy way to tell from action graph whether a given maniplex is a polytope?

E.g. if an i-edge has both end vertices in the same J-face and $i \notin J$, the maniplex is not a polytope.

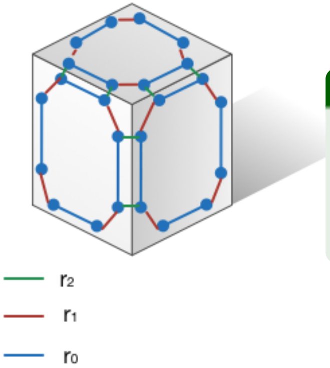
Oriented maps as Action Graphs



Example

An oriented map M is an action graph with signature $(1, 1)$. The usual expression $M = (D; R, L)$ of an oriented map due to Edmonds, in terms of darts, rotation, and dart-reversing involution, is the same as $M = (D; R; r)$.

Maps as Action Graphs



Example

A map M is an action graph with signature $(0, 3)$, i.e. $M = \Gamma(\Phi, \emptyset; r_0, r_1, r_2)$. The permutations r_0, r_1, r_2 and $r_0 r_2$ are fixed-point free involutions.

Hypermaps as Action Graphs

Example

A three-involution representation of a hypermap is an action graph with signature $(0, 3)$.

Oriented maniplexes as action graphs

Recall that a maniples is an action graph with signature $(0, n)$. The vertices are flags and the n involutions represent exchange maps on the flags of the polytope:
$\Gamma = (\Phi; \emptyset, \{r_0, r_1, \ldots, r_{n-1}\}), r_i r_j = r_j r_i, |i-j| > 2.$
A maniples is *orientable* if the underlying graph of Γ is bipartite. In this case the flags fall into two classes Φ_- and Φ_+. Define the following action graph:
$\Gamma_{+-} = (\Phi; r_{n-2} r_{n-1}, \{r_0 r_{n-1}, r_1 r_{n-1}, \ldots, r_{n-3} r_{n-1}\}), r_i r_j = r_j r_i, |i-j| > 2..$
The action graph Γ_{+-} is disconnected. We may distinguish the two connected components: Γ_+ and Γ_-. Hence we have:
$\Gamma_+ = (\Phi_+; r_{n-2} r_{n-1}, \{r_0 r_{n-1}, r_1 r_{n-1}, \ldots, r_{n-3} r_{n-1}\})$
$= (\Phi_+; R, \{s_0, s_1, \ldots, s_{n-3}\}), s_i s_j = s_j s_i, |i-j| > 2$

Morphisms of Action Graphs

A *morphism* of action graphs $\vartheta\colon \Gamma_1 \to \Gamma_2$ is defined for a pair Γ_1, Γ_2 provided that both action graphs have the same signature. The morphism ϑ is defined by a mapping $\vartheta\colon V_1 \to V_2$ between the corresponding sets of vertices of action graphs $\Gamma_1(V_1; \mathcal{R}_1; \varrho_1)$ and $\Gamma_2(V_2; \mathcal{R}_2; \varrho_2)$ such that for all $i \in I$ and all $j \in J$:

$$\begin{aligned} (\vartheta v)R_i &= \vartheta(vR_i), \\ (\vartheta v)r_j &= \vartheta(vr_j). \end{aligned} \tag{1}$$

According to widely used convention, morphisms have left action on the vertex set V.

A morphism for which $\vartheta\colon V_1 \to V_2$ is a bijection is called an *isomorphism*.

Automorphisms of Action Graphs

The morphism $\alpha\colon \Gamma(V;\mathcal{R};\varrho) \to \Gamma(V;\mathcal{R};\varrho)$ is an *automorphism* if the mapping $\vartheta\colon V \to V$ is a bijection.

The *group of automorphisms* $\operatorname{Aut}_{\mathrm{o}} \Gamma$ of the action graph consists of all *(edge-colour-preserving)* automorphisms of Γ. In other words, $\operatorname{Aut}_{\mathrm{o}} \Gamma$ is the centraliser of the corresponding monodromy group $\operatorname{Mon} \Gamma$ in the symmetry group $\operatorname{Sym}(V)$.

Important properties of automorphisms.

Theorem

The automorphism group of an action graph is the centraliser of the corresponding monodromy group in symmetry group over action graph vertices.

Let us note that in both cases the collections $\mathcal{R}$ and ϱ are taken as *ordered* collections.

Example

An oriented map M is an action graph with signature $(1, 1)$. The automorphism group $\mathrm{Aut_o}\, M$ corresponds with the group of orientation-preserving automorphisms. A map M (in three-involutory representation) is an action graph with signature $(0, 3)$. The automorphism group $\mathrm{Aut_o}\, M$ is the same as the automorphism group as defined in Jones and Thornton.

Extended Morphisms.

Let $\vartheta\colon V_1 \to V_2$ be a mapping. The morphism

$$\varepsilon = (\vartheta, \hat{i}, \hat{j}, K_1)\colon \Gamma(V_1; \mathcal{R}_1; \varrho_1, I_1, J_1) \to \Gamma(V_2; \mathcal{R}_2; \varrho_2, I_2, J_2)$$

is an *extended morphism*, if there exist mappings on index sets $\hat{\imath}\colon I_1 \to I_2$ and $\hat{j}\colon J_1 \to J_2$ and a subset $K_1 \subseteq I_1$ and $K_2 = \hat{i}(K_1)$ such that if $v \in V_1$, $i \in I_1$ and $j \in J_1$, then

$$\begin{aligned}(\vartheta v)R_{\hat{\imath}(i)} &= \vartheta(vR_i), \text{ if } i \notin K_1,\\ (\vartheta v)R^{-1}_{\hat{\imath}(i)} &= \vartheta(vR_i), \text{ if } i \in K_1.\\ (\vartheta v)r_{\hat{j}(j)} &= \vartheta(vr_j),\end{aligned} \tag{2}$$

An extended automorphism is *orientation preserving* if $K_1 = \emptyset$ and is *orientation reversing* if $K_1 = I_1$.

Extended Isomorphisms and automorphisms.

An extended morphism $\varepsilon = (\vartheta, \hat{i}, \hat{j}, K_1)$ is an *extended isomorphism* if and only if $\vartheta, \hat{i}, \hat{j}$ are bijections.
An extended isomorphism of an action graph to itself is an *extended automorphism*.

Automorphism groups.

The *group of extended automorphisms* $\mathrm{Aut}_{\mathrm{e}}\,\Gamma$ of the action graph consists of all automorphisms of Γ. Note that here we consider the collections $\mathcal{R}$ and ϱ to be just multisets.

Proposition

Let Γ be an action graph. Then

$$\mathrm{Aut}_{\mathrm{o}}\,\Gamma \leq \mathrm{Aut}_{\mathrm{e}}\,\Gamma \leq \mathrm{Aut}\,\Gamma,$$

where $\mathrm{Aut}\,\Gamma$ is the group of automorphisms of the underlying graph of Γ.

Note: we have to modify definitions, if Γ has multiple edges.

Extended Morphisms - revisided.

	orientation preserving	orientation reversing
color preserving	automorphism $\mathrm{Aut_o}\,\Gamma$	R maps to R^{-1}
color respecting	r_i to r_j or R_i to R_j, $i \neq j$	$\mathrm{Aut_e}\,\Gamma$ extended automorphism

In general, there are four groups of automorphisms possible!

Extended Automorphisms - The Case of Oriented Maps.

Example

Let Γ be an action graph of an oriented map $\Gamma(V;R;r)$. The morphism $\Gamma(V;R;r) \mapsto \Gamma(V;R^{-1};r)$ correspond to an orientation-reversing isomorphism; i.e. the operation of 'taking the mirror-image'.

Extended Automorphisms - The Case of Maps.

Example

The mappings $\Gamma(V; \emptyset; r_0, r_1, r_2) \mapsto \Gamma(V; \emptyset; r_2, r_1, r_0)$ which extends to automorphisms of action graphs corresponds to (self-)dualities of a map.

The Fundamental Lemma of Action Graphs.

Lemma (Fundamental Lemma of Action Graphs)

The action of automorphism group $\mathrm{Aut}_\mathrm{o}\,\Gamma(V;\mathcal{R};\varrho)$ *is semi-regular on* V.

Some Consequences.

The Fundamental Lemma of Action Graphs has several interesting consequences:

Corollary

a) $|\operatorname{Aut}_o \Gamma|$ *is a divisor of* $|V|$*;*

b) *The projection* $\Gamma \to \Gamma / \operatorname{Aut}_o \Gamma$ *is a regular covering projection;*

Definition

The quotient $T(\Gamma) = \Gamma \to \Gamma / \operatorname{Aut}_o \Gamma$ is an action graph called the *symmetry type graph*.

Regular Action Graphs.

The action graph Γ is *regular* if $\mathrm{Aut}_{\mathrm{o}}\,\Gamma(V;\mathcal{R};\varrho)$ acts regularly on V.

Proposition

$|\,\mathrm{Aut}_{\mathrm{o}}\,\Gamma(V;\mathcal{R};\varrho)| \leq |V|$, more precisely: $|\,\mathrm{Aut}_{\mathrm{o}}\,\Gamma(V;\mathcal{R};\varrho)|$ divides $|V|$ and equality is reached if and only if Γ is regular.

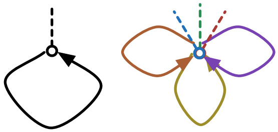
Corollary

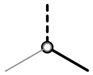
Let M be a oriented map on e edges. Then $|\,\mathrm{Aut}_{\mathrm{o}}(M)|$ divides $2e$ and M is regular if and only if $|\,\mathrm{Aut}_{\mathrm{o}}(M)| = 2e$.

Corollary

Let M be a map on e edges. Then $|\,\mathrm{Aut}_{\mathrm{o}}(M)|$ divides $4e$ and M is regular if and only if $|\,\mathrm{Aut}_{\mathrm{o}}(M)| = 4e$.

Regular Action Graphs are Cayley Graphs





Proposition

An action graph $\Gamma(V;\mathcal{R};\varrho)$ is regular if and only if it is a (colored Cayley graph).

Proposition

An action graph $\Gamma(V;\mathcal{R};\varrho)$ is regular if and only if its symmetry type graph $T(\Gamma(V;\mathcal{R};\varrho))$ is a 1-vertex graph.

Example

See symmetry type graphs of a generic Cayley graph, of a regular oriented map and of a regular map.

Symmetry type graphs with respect to subgroups.

Remark

It is possible to develop theory of G*-action graphs* Γ/G*, with respect to a subgroup of the full automorphism group* $\mathrm{G} \leq \mathrm{Aut}_{\mathrm{o}}\,\Gamma$*.*

Incidence Geometry.

An *incidence geometry of rank* r is a properly vertex-colored graph (Γ, c, T), where Γ is a graph and $c : V(\Gamma) \to T$ is a proper vertex coloring with $|T| = r$. Note that each vertex, alias geometry element, $v \in V(\Gamma)$ is of precise type $c(v) \in T$ and that only elements of different type may be incident: $u \sim v$ implies $c(u) \neq c(v)$.
Sometimes such a structure is called *pre-geometry* and for a geometry some further conditions are required.

Coset Geometry.

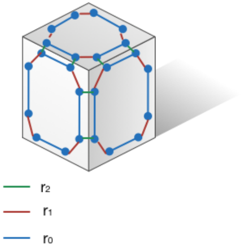
Let G be a group and let $H_1, H_2, \ldots, H_r$ be r of its subgroups. The structure $(G; H_1, H_2, \ldots, H_r)$ is called a *rank* r *coset geometry*.

Each coset geometry gives rise to an incidence geometry as follows. The vertices of G represents the right cosets $H_j a_i$ with two cosets being adjacent if and only if their intersection is disjoint. Furthermore, the coloring function c is given by $c(H_j a_i) = H_j$ (or, if we want to make it simple: $c(H_j a_i) = j$).

Coset Geometry and Regular Maps

Example

Any regular map gives rise to a coset geometry. The group G is generated by the three involutions $\langle r_0, r_1, r_2 \rangle$ and the subgroups corresponding to vertices, edges, and faces are generated by $\langle r_1, r_2 \rangle$, $\langle r_0, r_2 \rangle$, and $\langle r_0, r_1 \rangle$, respectively.



Action Geometry.

We may generalize the notion of coset geometries from groups to action graphs via Cayley graphs. Let $\Gamma(V; \mathcal{R}; \varrho)$ be an action graph and let us have $\mathcal{S}_i \subseteq \mathcal{R}$ and $\sigma_i \subseteq \varrho$ for $i = 1, 2, \ldots, r$. Then $\Delta_i = (V; \mathcal{S}_i; \sigma_i)$ is (possibly disconnected) action graph. By $F_i(\mathcal{S}_i, \sigma_i)$ we denote the partition of vertex set V into connected components (of Δ_i). The partition F_i corresponds to orbits of $\langle \mathcal{S}_i, \sigma_i \rangle$ on V. The elements of F_i will be called i-*faces* of $(\Gamma; \Delta_1, \Delta_2, \ldots, \Delta_r)$. The structure $(\Gamma; \Delta_1, \Delta_2, \ldots, \Delta_r)$ is called a rank r *action geometry*.
Using the same idea as with the coset geometries we my associate a rank r incidence geometry to each rank r action geometry.

From Coset Geometry to Action Geometry.

Example

Each non-trivial group G gives rise to a unique action graph $\Gamma(G) = (G, G - \{\mathrm{Id}\}, \emptyset)$ via the right regular representation. Note that in this case the action graph can be regarded as a particular Cayley graph for G. In the same way the coset geometry $(G; H_1, H_2, \dots, H_r)$ gives rise to the action geometry

$$(\Gamma(G); \Gamma(H_1), \Gamma(H_2), \dots, \Gamma(H_r)).$$

It is not hard to see that the underlying incidence geometries are isomorphic.

Remark

Several action geometries may give rise to the same incidence geometry.

Example

Let us take an action graph $M = (V; \emptyset; r_0, r_1, r_2)$, corresponding to a three-involutory representation of map. Then $F(r_1, r_2)$ corresponds to vertices of M, $F(r_0, r_2)$ corresponds to edges of M, and $F(r_0, r_1)$ corresponds to faces of M. The set of vertices of action graph, V is the set of flags of M. The corresponding incidence structure is displayed on Figure below.

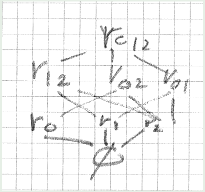


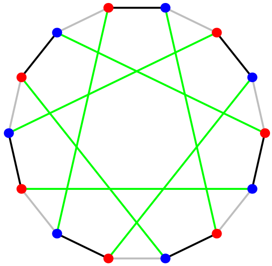
Figure :

Oriented Map as Action Graph.

Example

Let us take an action graph $M = (V; R; r)$ corresponding to an oriented map. The vertex set V then corresponds to set of darts of M, $F(R)$ corresponds to vertices of M and $F(r)$ corresponds to edges of M. The incidence structure over M correspond to a graph G, although we $(V; R; r)$ is well-defined *embedding* of G. We will show a standard technique used to define an action graph such that the corresponding geometry will employ faces of M as well. Take the action graph $M^* = (V; R, R^*; r)$ such that $R^* = R^{-1}r$. Then $F(R)$ correspond to the vertices, $F(r)$ correspond to the edges, and $F(R^*)$ correspond to the faces of M, respectively. The corresponding incidence geometry is now well-defined.

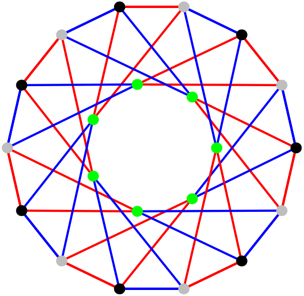
Trivalent graphs of Class 1 - Hypermaps



Example

Heawood graph is a bipartite trivalent graph. Any of its 3-edge-colorings gives rise to a hyper map.

Linegraphs of bipartite trivalent graphs as Action graphs



Example

The line graph of the Heawood graph is a quartic graph on 21 vertices. It inherits a 2-factorization composed of a blue and red triangular 2-factor. Any cyclic orientation of triangles gives rise to an action graph of signature (2,0). They all, in turn, give rise to the same incidence geometry - the Fano plane alias the Heawod graph with a given blue-red vertex coloring.

Constellations.

Example

Constellations of Lando and Zvonkin can be regarded as special action graphs. Namley an action graph $\Gamma = (\Phi; R_1, R_2, \ldots, R_n; \emptyset)$ is a *constellation* if $\prod_{i=1}^{n} R_i = \mathrm{Id}$.

Need for Cryptomorphism - Constellations

Sometimes the most concise choice of signatures for action graphs is not the best. By adding redundant permutations we may get a *cryptomorphic* description of essentially the same structure which may have richer extended group of automorphisms.

Example

If $\Gamma = (\Phi; R_1, R_2, \ldots, R_n; \emptyset)$ is not a constellation we may adjoin another permutation $R_{n+1} = (R_1 R_2 \ldots R_n)^{-1}$ and the action graph $\Gamma = (\Phi; R_1, R_2, \ldots, R_n, R_{n+1}; \emptyset)$ becomes a constellation.

Need for Cryptomorphism - Maps

Example

A general map may be described as $M = (V; \emptyset; r_0, r_1, r_2)$. It is easy to move from M to its dual map M^d using this model, but it is much more complicated to move from M to its Petrie dual. It is therefore easier to add an extra involution r_P and have $M' = (V; \emptyset; r_0, r_1, r_2, r_P)$ with additional requirement that $r_0 r_1 r_P = \mathrm{Id}$. These two definitions are cryptomorphic. Given M we may define $r_P = r_0 r_2$. In the same way M' defines back M. However, using M' it is easy to apply any of the six permutations on $\{0, 2, P\}$ to move from M to its dual, its Petrie dual, etc.

Need for Cryptomorphism - Oriented Maps

Example

(See Example 31.) In the model $(V; R, r)$ of oriented map the faces are not easily visible. We may define them as $F = R^{-1}r$. Sometimes it would be more convenient to use the model $(V; R, F, r)$ with additional requirement that $RF = r$. The dual map is simply obtained by swapping permutations R and F.

Maybe we should talk about categories and functors here.

From orientable maps to oriented maps

Theorem

Map M is orientable if and only if its flag-graph is bipartite.

Each orientable map $M = (\Phi; r_0, r_1, r_2)$ give rise to a pair of oppositely oriented maps $M_+ = (D_+, R, r)$ and $M_- = (D_, R, r)$.where D_+ contains all black flags of M and D_- contains all white flags of M. Furthermore $R = r_2 r_1, r = r_2 r_0$.

From oriented maps to orientable maps

Theorem

Each oriented map $M_+ = (D, R, r)$ *and its reverse* $M_- = (D, R^{-1}, r)$ *give rise to the same orientable map* $M = (\Phi, r_0, r_1, r_2)$

$\Phi = D_- \cup D_+, r_0(d_+) = r(d)_-, r_0(d_-) = r(d_+), r_1(d_+) = R(d)_-, r_1(d_-) = R^{-1}(d)_+, r_2(d_-) = d_+, r(d_+) = d_-$

Chiral maps in the sense of Conway

Definition

An oriented map (D, R, r) is chiral in the sense of Conway if it is not isomorphic (as oriented map) to its reverse (D, R^{-1}, r)

This definition carries over to an orientable map.

Definition

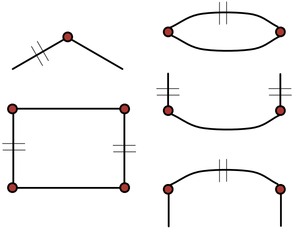
An orientable map M is chiral if the two oriented maps D_- and D_+ are not isomorphic (as oriented maps). in the sense of Conway if it is not isomorphic (as oriented map) to its reverse (D, R^{-1}, r)

Information Stored in Symmetry Type Graphs

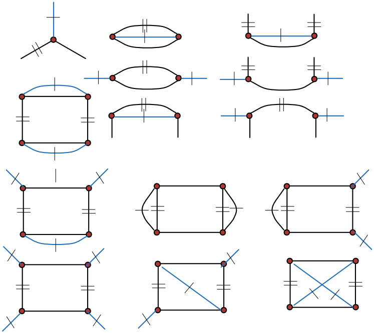
Theorem

Let $(\Gamma; \Delta_1, \Delta_2, \ldots, \Delta_r)$ *be an action geometry and let* $T(\Gamma)$ *be its symmetry type graph. The number of connected components of* $T(\Delta_i)$ *denotes the number of orbits of* $i - faces$ *of* Γ*.*

ctually, we have to define also J-flags, where J is a subset of types. Then we can tell when a map is vertex-, edge-, or face-transitive. Using this notation we may describe examples of maps that are vertex- and edge-transitive but are not arc-transitive. Using indices, we get the reverse correspondence: vertex-transitive is 12-transitive, edge-transitive is 02-transitive, face-transitive is 01-transitive.



Edge-transitive maps are very well understood. It is well-known they come in 14 types.. (Graver, Watkins; Širan, Tucker, Watkins; Orbanić. etc.). There are five possible quotients of an edge quadrangle.



Edge-transitive maps are very well understood. It is well-known they come in 14 types. There are five possible quotients of an edge. By adding all possible 1-edges to each quotient of the edge quadrangle we obtain the 14 types.

Strong and weak Edge-Transitive Oriented Maps

The vertices of the action graphs and symmetry type graphs of maps are flags while the vertices of action graphs and symmetry type graphs of oriented maps are darts.
An oriented map may be edge-transitive in the *strong sense* if the group of orientation preserving automorphisms acts transitively on the edges. On the other hand, if the map is edge-transitive as a map but not as an oriented map, then we say that it is edge-transitive in the *weak sense*.

Edge-transitive oriented maps.

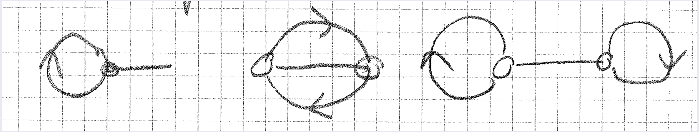
Proposition (Karabaš, Nedela)

There are 8 symmetry type graphs $(8 = 3\{\textit{strong}\} + 5\{\textit{weak}\})$ admitting edge-transitive oriented maps.

Edge-transitive oriented maps.

Proposition

An oriented map Γ is edge-transitive in the strong sense if and only if $T(\Gamma)$ is one of the following



Edge-transitive oriented maps proof.

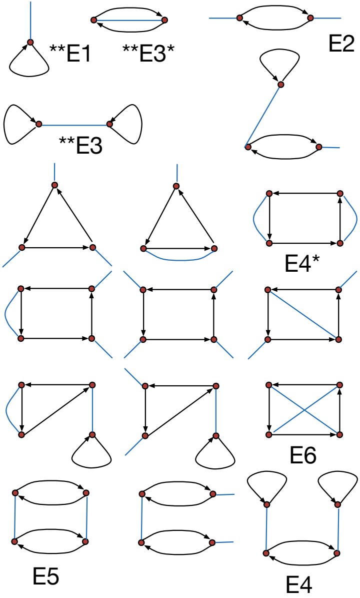
Lemma

The action graph Γ is edge-transitive in the strong sense if and only if the subgraph of $T(\Gamma)$ induced by involution r is connected.

Lemma

The action graph Γ is edge-transitive in the weak sense if and only if the subgraph of $T(\Gamma)$ induced by involutions is connected, or if there exist an extended involution on $T(\Gamma)$ exchanging the two edges corresponding to r.

Small oriented symmetry type graphs.

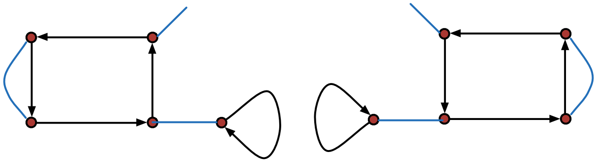


All oriented k-orbit symmetry type graphs (of oriented maps) for $k = 1, 2, 3, 4$. (**)
Edge-transitive in the strong sense: 3 cases E1, E3, E3*.
() Edge-transitive in the weak sense: 5 cases: E2, E4, E4*, E5, E6.

Exercise

Determine dual pairs of oriented symmetry types. Hint: among the edge-transitive ones there are two dual pairs.

Smallest chiral symmetry type graph.



A *chiral* symmetry type graph is not isomorphic to its reverse. Hence any oriented map of this type is chiral (in the sense of Conway).

Thanks for your attention.