# On Skew-Homomorphisms

B. Kuzma[1] G. Dolinar G. Nagy P. Szokol

[1]UP FAMNIT

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# Skew-morphism

Given a function $\Phi\colon \mathcal{A} \to \mathcal{A}$, let

- $\Phi^0 = \mathrm{Id}$.
- $\Phi^k(x) = \Phi(\Phi^{k-1}(x))$.
- $\Phi^{-1}$ the inverse of $\Phi$.

## Definition

$\Phi\colon \mathcal{A} \to \mathcal{A}$ is *skew-morphism* if for some function $\kappa\colon \mathcal{A} \to \mathbb{N}$

$$\Phi(ab) = \Phi(a)\Phi^{\kappa(a)}(b)$$

## Remark

*For $\Phi$ bijective can also take $\kappa\colon A \to \mathbb{Z}$.*

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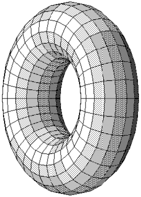

- A map $M$ is a 2-cell embedding of a simple connected graph $\Gamma$ into oriented surface.
  - $M$ is Cayley if exists subgroup $A \leq \mathrm{Aut}^+(M)$ acting regularly on $V(\Gamma)$.
  - $M$ is regular if $\forall (a, b), (c, d) \in E(\Gamma)$ there is $\Phi \in \mathrm{Aut}^+(M)$ such that $(\Phi(a), \Phi(b)) = (c, d)$.

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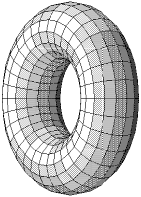

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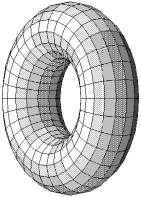

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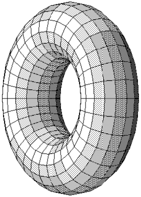

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Cyclic extensions of groups!

- $A, C \leq G$ subgroups with $A \cap C = 1$ and $G = AC$. Assume $C = \langle c \rangle$.
  - Given $g = ac \in G$, we have

$$ca = \Phi(a)c^i \qquad \text{for some } i \in \mathbb{Z}$$

Then,

$$\begin{aligned} c(ab) &= \Phi(ab)c^k \\ &= (ca)b = \Phi(a)c^i b = \Phi(a)\Phi^i(b)c^t \end{aligned}$$

  - By uniqueness, $c^k = c^t$ and

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# Case $\kappa(G) = 0$

## Lemma

*$\Phi\colon M_n(\mathbb{F}) \to M_n(\mathbb{F})$ skew-morphism.*
*If $\kappa(G) = 0$ for some $G \in \mathrm{GL}_n(\mathbb{F})$ then*

$$\Phi(X) = MX.$$

## Proof.

$$\Phi(X) = \Phi(G \cdot G^{-1}X) = \Phi(G)\Phi^{\kappa(G)}(G^{-1}X) = \Phi(G) \cdot G^{-1}X.$$

Define $M := \Phi(G)G^{-1}$. □

# Case Φ linear.

**Theorem**

*$\Phi\colon M_n(\mathbb{F}) \to M_n(\mathbb{F})$ linear skew-morphism.*
*Then exists $s \in \mathbb{N}$ and $\lambda \in \mathbb{F}$ such that*

$$\Phi(X) = MXN. \qquad \text{where } N \in \mathrm{GL}_n(\mathbb{F}) \text{ and } N^{1-s} = NM^s = \lambda I.$$

**Corollary**

*$\Phi\colon M_n(\mathbb{F}) \to M_n(\mathbb{F})$ linear, unital skew-morphism.*
*Then*

$$\Phi(X) = N^{-1}XN.$$

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# General surjective $\Phi$

## Example

There exists a nonlinear unital, bijective skew-morphism $\Phi\colon M_2(\mathbb{Z}_2) \to M_2(\mathbb{Z}_2)$.

$$
\begin{array}{ll}
\phi\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\right)=\begin{pmatrix}0&1\\1&1\end{pmatrix},\ \kappa\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\right)=2\,, & \phi\left(\begin{pmatrix}0&1\\1&1\end{pmatrix}\right)=\begin{pmatrix}1&1\\1&0\end{pmatrix},\ \kappa\left(\begin{pmatrix}0&1\\1&1\end{pmatrix}\right)=3\,,\\
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## Theorem

$\Phi\colon M_n(\mathbb{F}) \to M_n(\mathbb{F})$ *surjective skew-morphism. Then*

- $\operatorname{rk} \Phi(X) = \operatorname{rk} X$.
- *If* $\kappa(\mathrm{GL}_n) \geq 1$ *and* $\kappa(G) > 1$ *for some* $G \in \mathrm{GL}_n(\mathbb{F})$ *THEN* $\Phi^s = \mathrm{id}$ *for some* $s \geq 1$.
- *Assume* $\kappa(\mathrm{GL}_n) = \{1\}$. *Then,*

$$\Phi(X) = \begin{cases} S^{-1} X_\sigma S, & X \in \mathrm{GL}_n \\ \gamma S^{-1} X_\sigma G, & X \in M_n \backslash \mathrm{GL}_n, \end{cases} \qquad (n \geq 3)$$

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$$\Phi(X) = \begin{cases} \gamma S^{-1}\mathrm{Cof}(X_\sigma)G, & X \in \mathrm{GL}_n \\ \gamma S^{-1}\mathrm{Cof}(x_\sigma)\mathrm{Cof}^s(f_\sigma^t)G, & X = xf^t \in M_n \backslash \mathrm{GL}_n, \end{cases}$$

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# Proofs: Case $\Phi$ linear.

**Theorem**

$\Phi\colon M_n(\mathbb{F}) \to M_n(\mathbb{F})$ *linear skew-morphism. Then* $\Phi(X) = MXN$.

**Proof.**

WLOG $\kappa(G) \geq 1$ for each $G \in \mathrm{GL}_n(\mathbb{F})$.

(i) Assume $0 \neq A \in \operatorname{Ker}\Phi$.

- Take any rank-one $R = xf^t$.
- Exists invertible $S$ and rank-one $T$ such that

$$R = SAT.$$

- Hence, $\kappa(S) \geq 1$, so

$$\Phi(R) = \Phi(S)\Phi^{\kappa(S)}(AT) = \Phi(S)\Phi^{\kappa(S)-1}(\Phi(A)\Phi^{\kappa(A)}(T))$$
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- So, $\Phi(X) = 0 = 0 \cdot X \cdot I$. □

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WLOG $\kappa(G) \geq 1$ for each $G \in \mathrm{GL}_n(\mathbb{F})$.

(ii) Assume $\mathrm{Ker}\,\Phi = 0$. Then:

- $\Phi$ bijective.
- $\Phi(\mathrm{GL}_n) \subseteq \mathrm{GL}_n$.
- Hence, $\Phi^{-1}(\mathrm{Sing}_n) \subseteq \mathrm{Sing}_n$. By Dieudonné

$$(i)\ \Phi(X) = MXN \quad \text{or} \quad (ii)\ \Phi(X) = MX^tN. \qquad \square$$

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- $\Phi(\mathrm{GL}_n) \subseteq \mathrm{GL}_n$.
  - By surjectivity $\exists B \in M_n$ such that $\phi(B) = I$. Hence,

$$I = \phi(B) = \phi(IB) = \phi(I)\phi^{\kappa(I)}(B).$$

So, $\phi(I)$ invertible. Then, for $A \in \mathrm{GL}_n$:

$$\phi(I) = \phi(AA^{-1}) = \phi(A)\phi^{\kappa(A)}(A^{-1})$$

and $\phi(A)$ is also invertible. So $\phi(\mathrm{GL}_n) \subseteq \mathrm{GL}_n$.

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- Each homomorphism is also a skew-morphism. On $M_n(\mathbb{F})$ they take three forms:
  (i) $\Phi(X) = f(\det X)S^{-1}X_\sigma S$ or
  (ii) $\Phi(X) = f(\det X)S^{-1}\mathrm{Cof}(X_\sigma)S$ or
  (iii) are degenerate.
- Our approach would classify those unital skew-morphisms on $\mathrm{GL}_n$ with extensions to $M_n$.