Equistarable Bipartite Graphs

Nina Chiarelli

Joint work with Endre Boros and Martin Milanič

UP Famnit, Koper, May 2015

Outline

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Equistable graphs

Payan, 1980

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Equistable graphs

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Definition

A graph G is said to be **equistable** if there exists a mapping $\varphi : V \to [0,1]$ such that for all $S \subseteq V$,

$$S \text{ is a maximal stable set} \iff \varphi(S) := \sum_{v \in S} \varphi(v) = 1.$$

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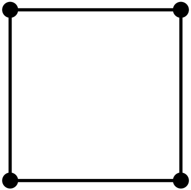
Such a φ is called an **equistable weight function** of G.

Example of an equistable graph



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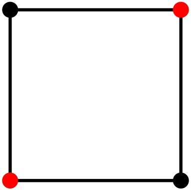

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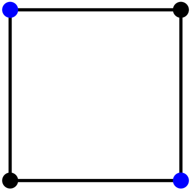

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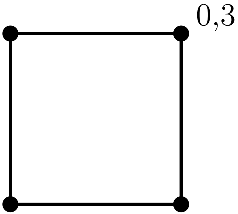

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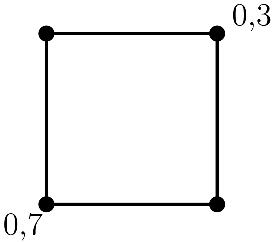

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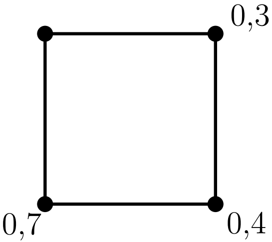

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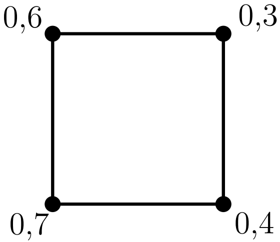
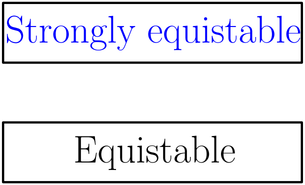


Equistable graphs

in connection to some other graph classes

Equistable graphs
in connection to some other graph classes

Mahadev et al., 1994



Equistable graphs

in connection to some other graph classes

Strongly equistable

Equistable

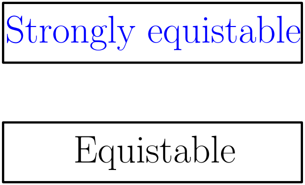
Mahadev et al., 1994

Definition

Given a graph $G = (V, E)$, let $\mathscr{S}(G)$ be the set of all maximal stable sets of G, and $\mathscr{T}(G)$ the set of all other nonempty subsets of $V(G)$.

Equistable graphs

in connection to some other graph classes



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Given a graph $G = (V, E)$, let $\mathscr{S}(G)$ be the set of all maximal stable sets of G, and $\mathscr{T}(G)$ the set of all other nonempty subsets of $V(G)$. A graph is said to be strongly equistable if for each $T \in \mathscr{T}(G)$ and for aech $\gamma \leq 1$,

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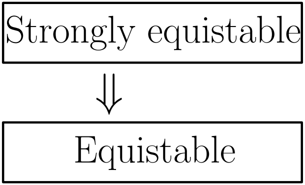

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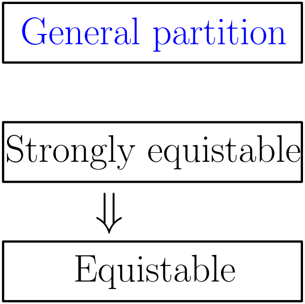

DeTemple et al., 1989

Equistable graphs
in connection to some other graph classes

General partition

Strongly equistable

⇓

Equistable

DeTemple et al., 1989

Theorem (McAvaney et al., 1993)

A graph G is a general partition graph if and only if every edge of G is contained in a strong clique.

Equistable graphs
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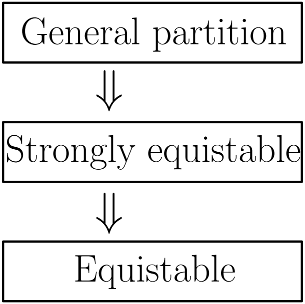

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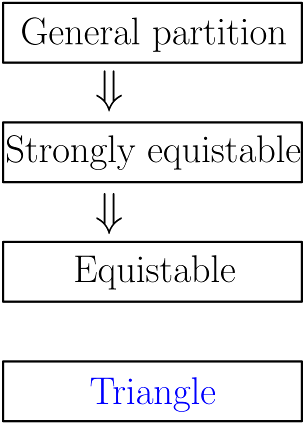

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General partition

⇓

Strongly equistable

⇓

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Triangle

McAvaney et al., 1993

Triangle condition

For every maximal stable set S in $G = (V, E)$ and every edge uv in $G - S$ there is a vertex $s \in S$ such that $\{u, v, s\}$ induces a triangle in G.

Equistable graphs

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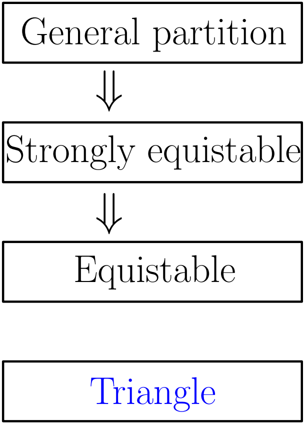

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Graphs satisfying this condition are called triangle graphs.

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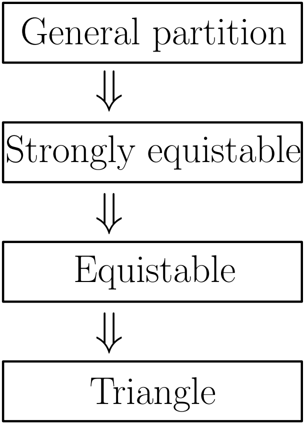

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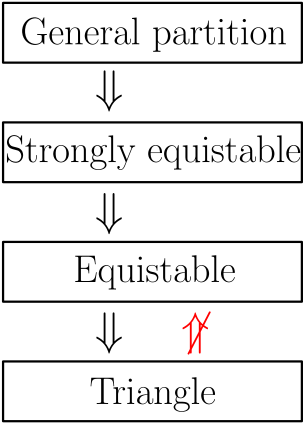

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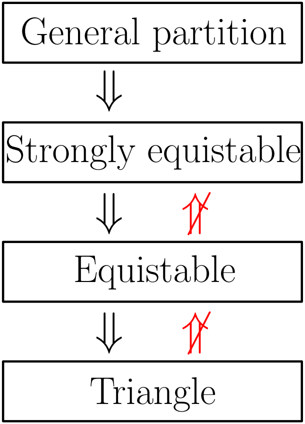

Equistable graphs

in connection to some other graph classes

General partition

$\Downarrow$ $\not\Uparrow$

Strongly equistable

$\Downarrow$ $\not\Uparrow$

Equistable

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Triangle

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Milanič, Trotignon, 2014

Given a graph G and a vertex $v \in V(G)$, the **star rooted at** v is the set $E(v)$ of all edges incident with v.

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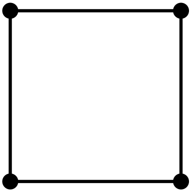

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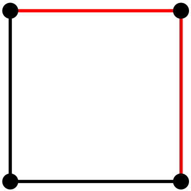

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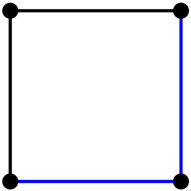

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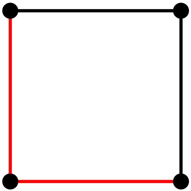

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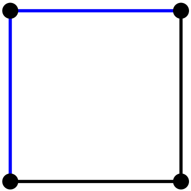

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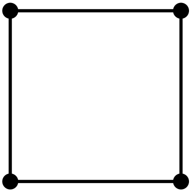

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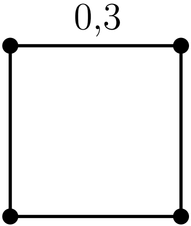

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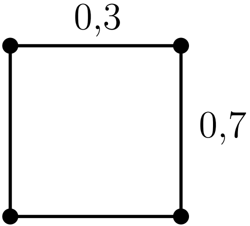

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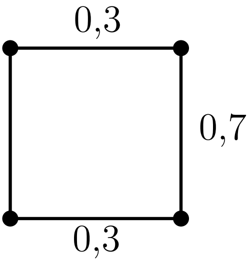

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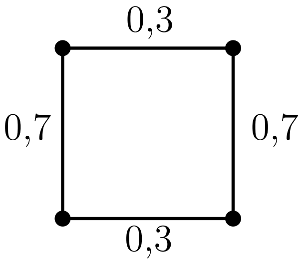
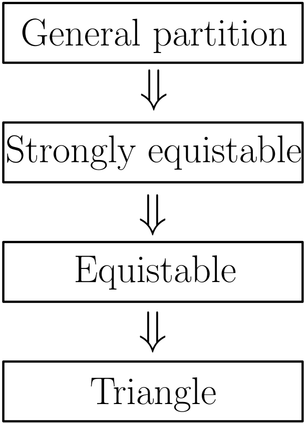


Equistarable graphs

in connection to some other graph classes



Equistarable graphs

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$\overline{L(G)}$

General partition

$\Downarrow$

Strongly equistable

$\Downarrow$

Equistable

$\Downarrow$

Triangle

Equistarable graphs
in connection to some other graph classes

G

$\overline{L(G)}$

General partition

$\Downarrow$

Strongly equistable

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Equistable

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Equistarable graphs
in connection to some other graph classes

triangle-free graph G | $\overline{L(G)}$

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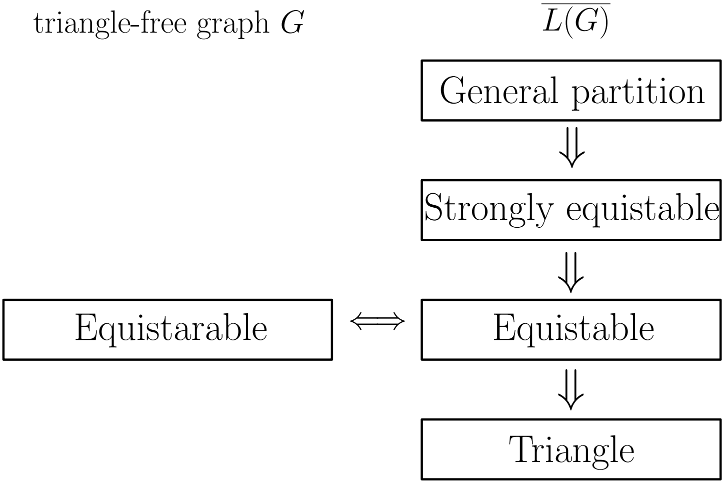

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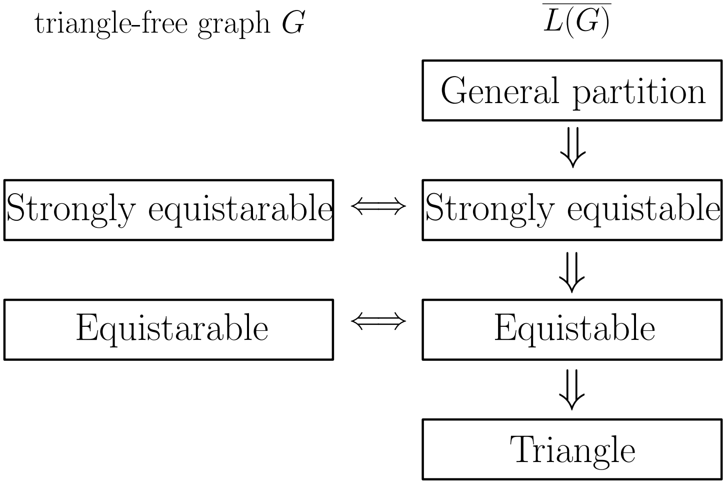

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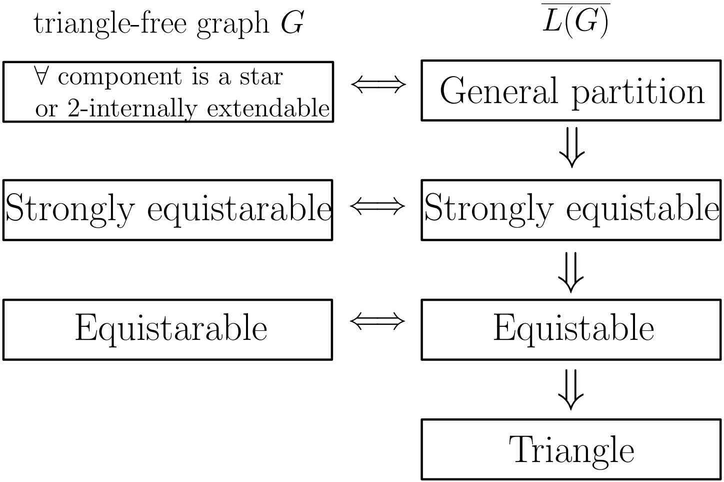

Recall: A graph G is 2-**extendable** if it is connected, contains a 2-matching and every 2-matching extends into a perfect matching.

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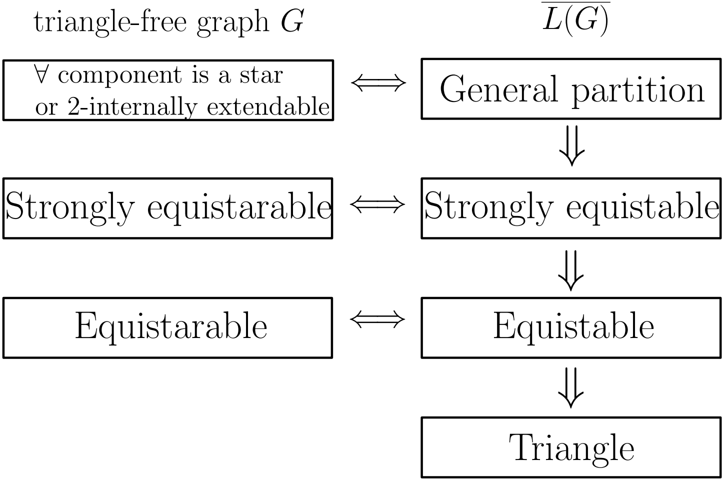
Definition

A graph is 2-**internally extendable** if every 2-matching can be extended to a perfect internal matching.


triangle-free graph G
$\overline{L(G)}$
∀ component is a star or 2-internally extendable
⟺
General partition
⇓
Strongly equistarable
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Strongly equistable
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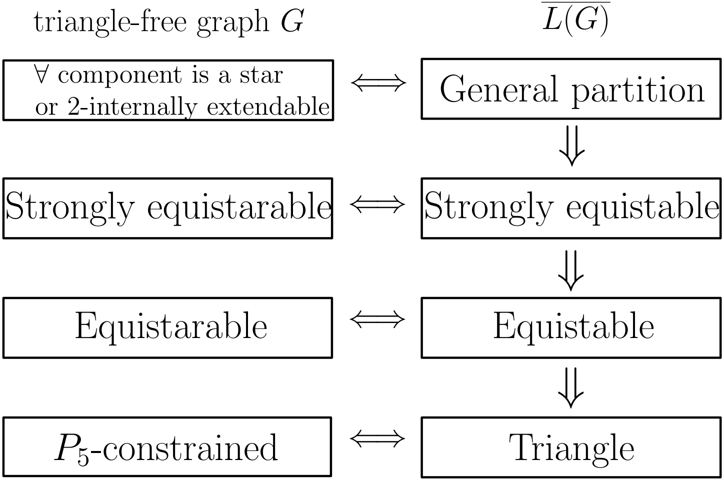
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Definition

A graph is P_5-**constrained** if every vertex of degree 2 is not a central vertex of a P_5.

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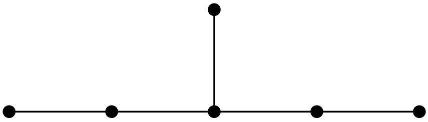

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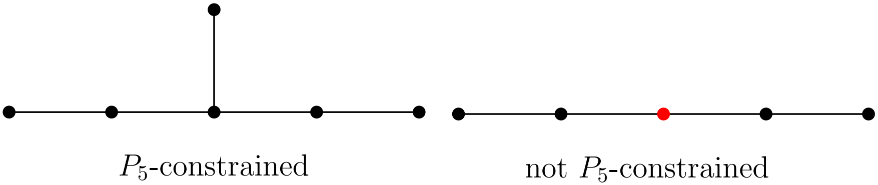
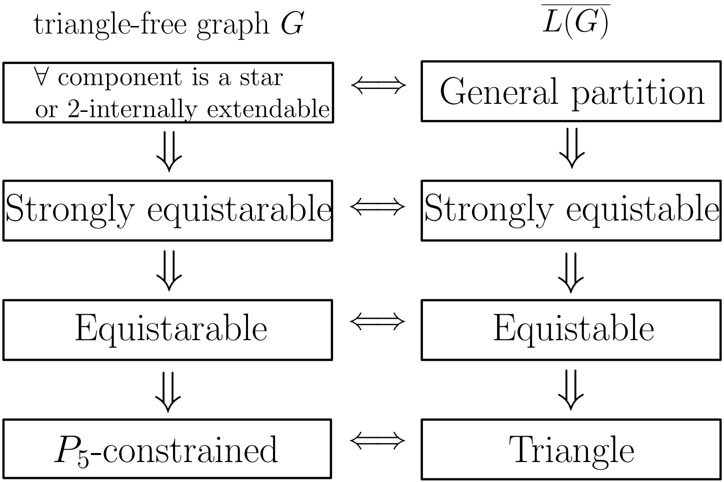


triangle-free graph G		$\overline{L(G)}$
$\forall$ component is a star or 2-internally extendable	$\Longleftrightarrow$	General partition
		$\Downarrow$
Strongly equistarable	$\Longleftrightarrow$	Strongly equistable
		$\Downarrow$
Equistarable	$\Longleftrightarrow$	Equistable
		$\Downarrow$
P_5-constrained	$\Longleftrightarrow$	Triangle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Outline

Bipartite graphs

A graph is **bipartite** if its vertex set can be partitioned into two stable sets.

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Theorem

For a bipartite graph G the following are equivalent:
(a) Every connected component of G is either a star or 2-internally extendable.
(b) G is strongly equistarable.
(c) G is equistarable.

Proof sketch

Since bipartite graphs are triangle-free, we know (a)$\Rightarrow$(b)$\Rightarrow$(c).

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Lemma

Let G be a connected equistarable bipartite graph with $\delta(G) \geq 2$. Then, G is 1-extendable.

Proof sketch

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Lemma

Let G be a connected equistarable bipartite graph with $\delta(G) \geq 2$. Then, G is 1-extendable.

Theorem (Plummer)

Let $k \geq 1$ and let $G = (V, E)$ be a connected bipartite graph with a bipartition $\{A, B\}$ of its vertex set and $V \geq 2k$. Then, G is k-extendable if and only if $|A| = |B|$ and for all non-empty subsets $X \subseteq A$ with $|X| \leq |A| - k$, it holds that $|N(X)| \geq |X| + k$.

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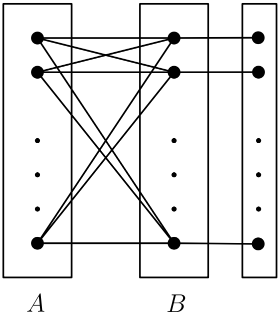

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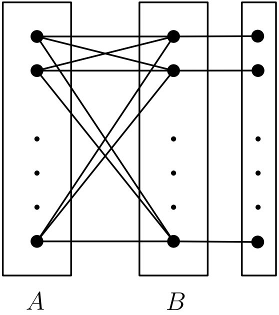

Suppose:
$|A| = k$
$|B| = l$

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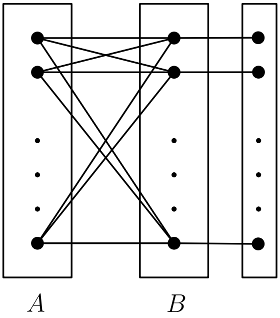

Suppose:
$|A| = k$
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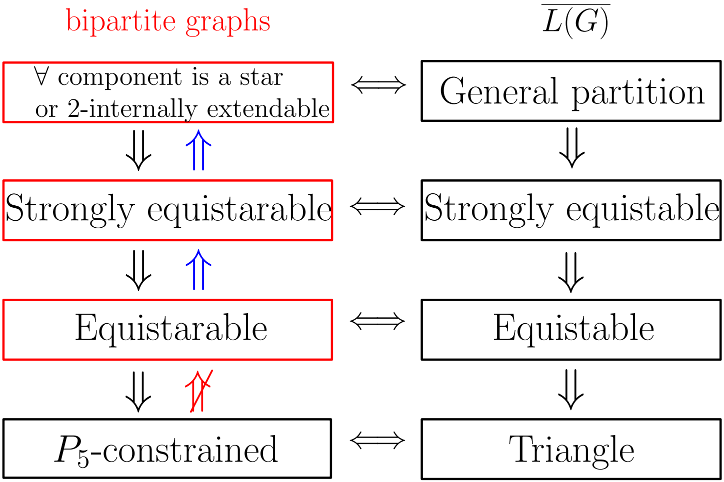
Every such graph with
$3 \leq l \leq k+1$
is not 2-internally extendable.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Forests

A **forest** is an acyclic graph.

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Theorem

For every forest F the following are equivalent:
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(b) F is strongly equistarable.
(c) F is equistarable.
(d) F is P_5-constrained.

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Lemma

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Let F be P_5-constrained. We can assume that F is connected and not a star.

Fix a 2-matching $M = \{e, f\}$, and consider the (unique) shortest path P between e and f.

We construct another matchig M' by putting in for every vertex of P, not covered by M, an arbitrary edge incident with it and not in P.
(Since F is P_5-constrained, all the vertices of P have degree ≥ 3.)

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What we have left is a forest F' consisting of some nontrivial trees, each of which contains at most one edge of $M' \cup M$. By the previous lemma matching $M' \cup M$ can be extended to a perfect internal matching of F.

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Therefore, every connected component of F is either a star or 2-internally extendable.

Conclusion

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Open questions

- What is the complexity of recognizing equistarable graphs?
- Is every perfect equistable graph a general partition graph?

Thank you!