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Evropski socialni sklad

Coloring graphs without long induced paths

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Graph coloring

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- a **k-coloring** is an assignment of numbers $\{1, 2, \ldots, k\}$ to the vertices such that any two adjacent vertices receive distinct numbers

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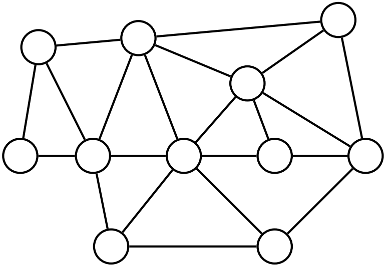

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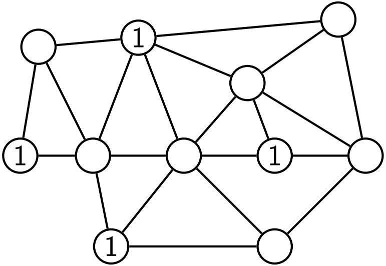

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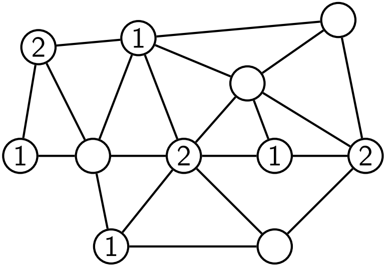

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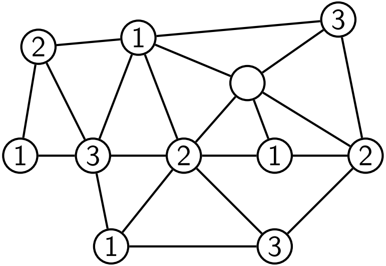

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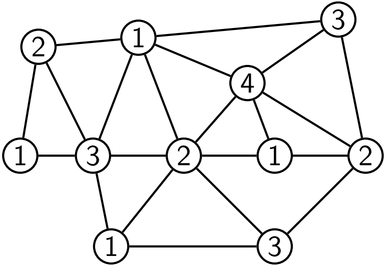

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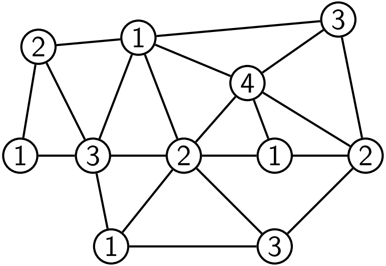

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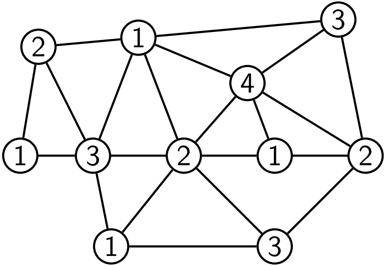

- the related decision problem is called **k-colorability**
- it is NP-complete for every $k \geq 3$

k-colorability in H-free graphs

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Theorem (Lozin & Kaminski 2007)

Let H be any graph that is not the disjoint union of paths. Then k-colorability is NP-complete in the class of H-free graphs, for all $k \geq 3$.

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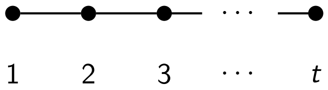

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For fixed k, the k-colorability problem is solvable in polynomial time in the class of P_5-free graphs.

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Open Problem

Determine the complexity of 4-colorability for P_6-free graphs.

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Is there any t such that 3-colorability is NP-hard for P_t-free graphs?

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- that leaves a 2-SAT problem, which can be solved efficiently

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- compute a vertex subset of constant size whose second neighborhood is the whole graph [Camby and S. 2014]

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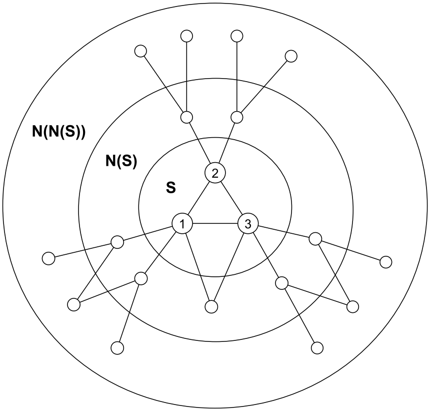

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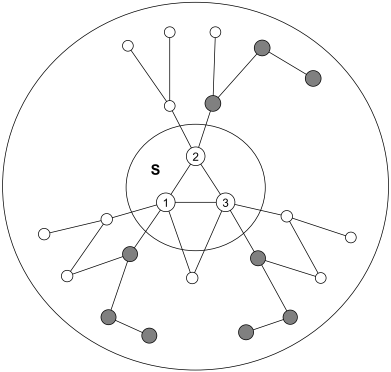

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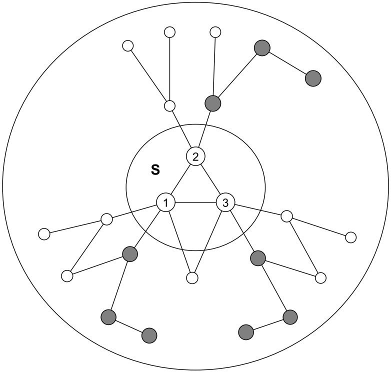

- this lets the seed grow, and the number of vertices that have only two colors left on their list

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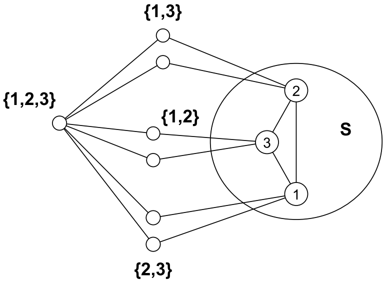

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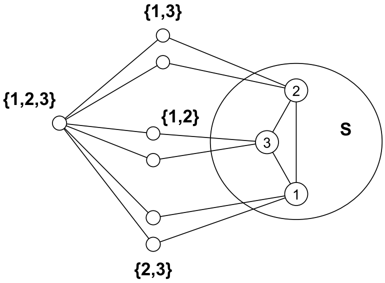

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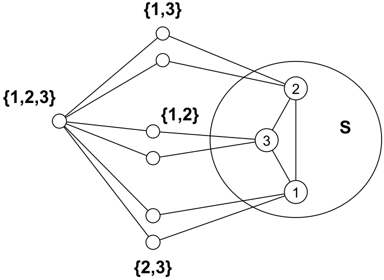

- after substituting each palette by $O(n^{10})$ new ones, we can get rid of these vertices
- then we solve the $O(n^{30})$ 2-SAT problems

Obstructions against 3-colorability

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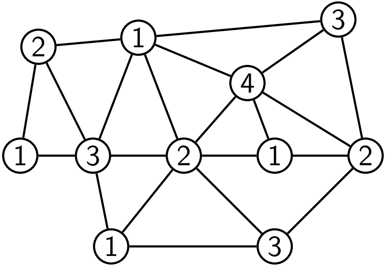

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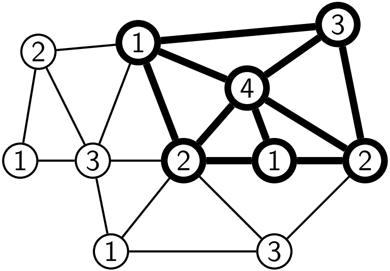

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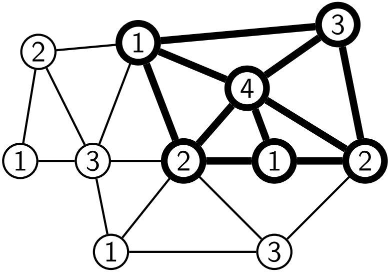

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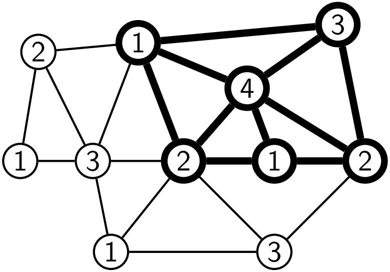

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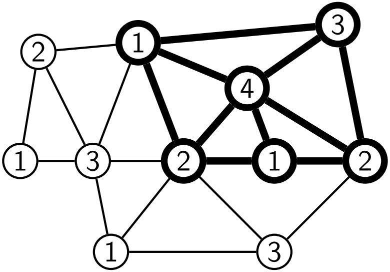

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- certifying coloring algorithm: output either a coloring or a small obstruction

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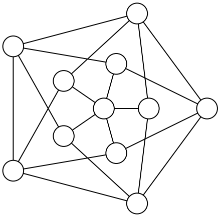

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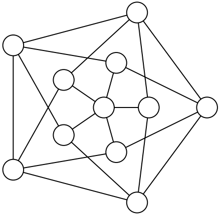

Theorem (Bruce, Hòang & Sawada 2009)

There are six obstructions in the class of P_5-free graphs.

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There are 24 obstructions in the class of P_6-free graphs.

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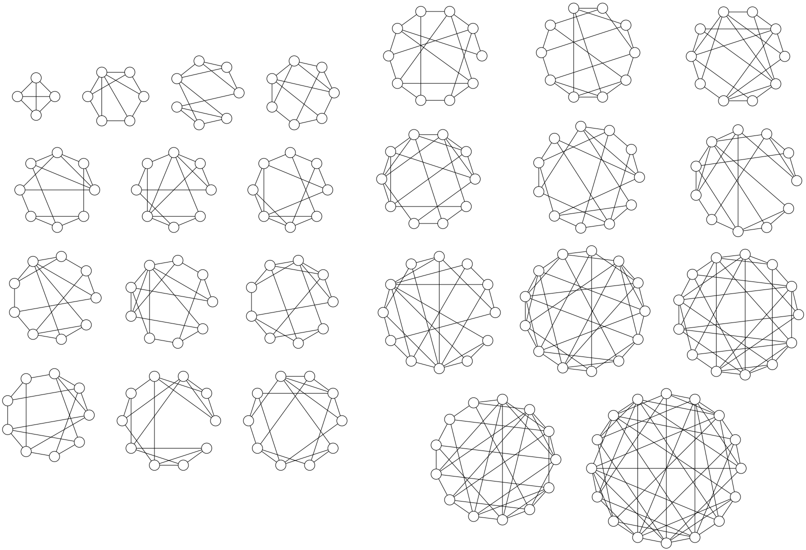
Theorem (Chudnovsky, Goedgebeur, S. & Zhong 2015)

There are 24 obstructions in the class of P_6-free graphs.

Moreover, if H is connected and not a subgraph of P_6, there are infinitely many obstructions in the class of H-free graphs.



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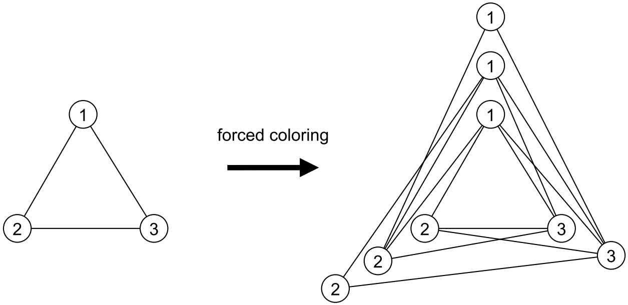

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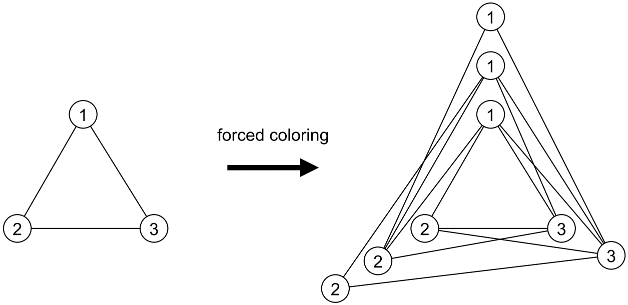

- the colored subgraph we call a **maximal tripod**

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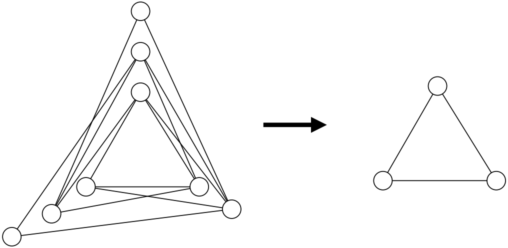
1. Prove that contracting a maximal tripod to a triangle is safe



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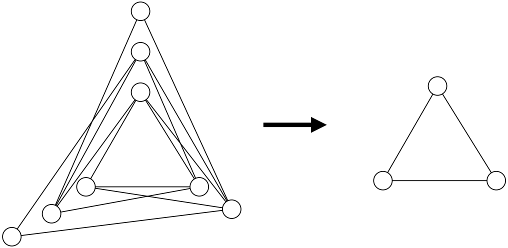

2. Prove the theorem for $(P_6, \text{diamond})$-free graphs
   - Use an automatic proof, building on a method of Hòang et al.
   - Exhaustive enumeration, exploiting properties of minimally non-3-colorable graphs

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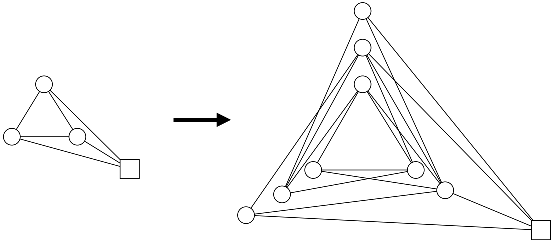

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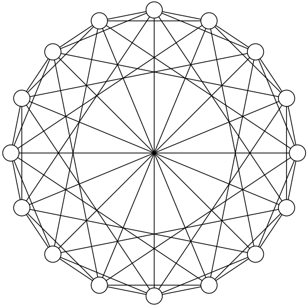

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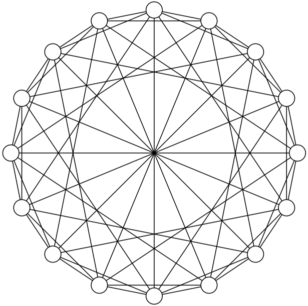

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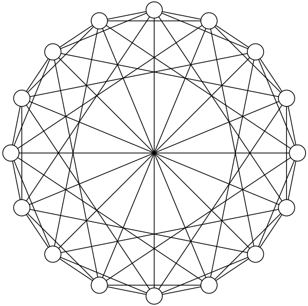

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