The Graph Bicycle Spectrum

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A matroid M is an ordered pair $(E, \mathcal{I})$ where E is a finite set and $\mathcal{I}$ is a collection of subsets of E satisfying the following three conditions:

(I1) $\emptyset \in \mathcal{I}$.

(I2) If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$.

(I3) If $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$, then there is an element e of $I_2 - I_1$ such that $I_1 \cup e \in \mathcal{I}$. *(Exchange Property)*

The members of $\mathcal{I}$ are called the independent sets of M and E is called the ground set of M. Any subset of E that is not independent is called dependent.

A matrix example

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Label the columns of this matrix with a, b, c, d, e, f.
Interpreting independence in the standard way, we have a matroid on the ground set $\{a, b, c, d, e, f\}$

A matroid M can also be defined by its set of minimal dependent sets called circuits. The set of circuits of M is denoted by $\mathcal{C}$ or $\mathcal{C}(M)$.

(C1) $\emptyset \notin \mathcal{C}$.

(C2) If $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$.

(C3) If C_1 and C_2 are distinct members of $\mathcal{C}$ and $e \in C_1 \cap C_2$, then there is some $C_3 \in \mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) - e$. *(Circuit Elimination Axiom)*

A graphic example

Given a graph (V, E), we define the cycle matroid $M(G)$:

- The ground set of $M(G)$ is E.
- $C \subseteq E$ is a circuit of $M(G)$ if and only if C is a cycle in G.

A matroid which can be realized as the cycle matroid of some graph is called graphic.

Dual Matroids

A maximal independent set of a matroid M is called a basis of M.
A matroid is well defined by specifying its bases, $\mathcal{B}(M)$.
Let M be a matroid on ground set E. Then the dual matroid of M, denoted M^*, is the matroid on E with bases $\{E - B : B \in \mathcal{B}(M)\}$.

Duals of graphic matroids

Theorem (Tutte)

If $M = M(G)$ is a graphic matroid, then M^* is graphic if and only if G is planar.

If M is the matroid of a planar graph G, then M^* is the matroid of the planar dual G^*.

$$M^*(G) = M(G^*)$$

Matroid M	**Dual matroid** M^*
basis B	basis complement $E - B'$
basis complement $E - B$	basis B'
circuit	cocircuit (bond)
cocircuit (bond)	circuit
hyperplane H	circuit complement $E - C'$
hyperplane comp. $E - H$	circuit C'

A hyperplane is a maximal non-spanning set. The circuits of M^* are the hyperplane complements of M.

Designs

A balanced incomplete block design $B(v, b, r, k, \lambda)$ is a pair $(X, \mathcal{B})$ such that

- $|X| = v, |B| = b$
- $\forall B \in \mathcal{B}, B \subseteq X$ and $|B| = k$
- $\forall x \in X, |\{B \in \mathcal{B} : x \in B\}| = r$
- $\forall x, y \in X, |\{B \in \mathcal{B} : \{x, y\} \subseteq B\}| = \lambda$

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Example:
The Fano Plane is a $(9, 7, 3, 3, 1)$ balanced incomplete block design.
It is also a (geometric) matroid.

Designs

For integer $t > 1$, a t-design t-(v, k, λ) is is a pair $(X, \mathcal{B})$ such that

- $|X| = v$
- $\forall B \in \mathcal{B}, B \subseteq X$ and $|B| = k$
- $\forall Y \subseteq X$ with $|Y| = t, |\{B \in \mathcal{B} : Y \subseteq B\}| = \lambda$

Designs

A matroid design (or equicardinal matroid) is a matroid whose hyperplanes all have the same size.
A perfect matroid design (PMD) is a matroid whose flats of each specific rank all have the same size.

The flats, or the circuits, or the independent sets of a PMD form a t-design (Young & Edmonds, early 1970s)

Matroid M	**Dual matroid** M^*
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hyperplane H	circuit complement $E - C'$
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Circuit spectrum

The circuit spectrum of a matroid M is

$$spec(M) = \{|C| : C \in \mathcal{C}(M)\}$$

Analagous to the well-studied cycle spectrum of a graph,

$$spec(G) = \{|C| : C \in \mathcal{C}(G)\}$$

where $\mathcal{C}(G)$ is the collection of all cycles in graph G.

Subdivisions

- A subdivision of a matroid is obtained by replacing each element by a series class.
- In a graphic matroid, this corresponds to replacing each graph edge by a path.

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- A subdivision of a matroid is obtained by replacing each element by a series class.
- In a graphic matroid, this corresponds to replacing each graph edge by a path.
- A k-subdivision is obtained by replacing each element by a series class of size k.
- In a graphic matroid, this corresponds to replacing each edge by a path of length k.

Small spectrum binary matroids

A matroid is binary provided it can be represented by (the columns of) a matrix with binary entries. **Every graphic matroid is binary.**

Theorem (Murty, 1971)

Let M be a connected binary matroid. For $\eta \in \mathbb{Z}^+$, spec$(M) = \{\eta\}$ if and only if M is isomorphic to one of the following matroids.

(i) *an η-subdivision of $U_{0,1}$*

(ii) *a k-subdivision of $U_{1,n}$, where $\eta = 2k$ and $n \geq 3$*

(iii) *an l-subdivision of $PG(r, 2)^*$, where $\eta = 2^r l$ and $r \geq 2$*

(iv) *an l-subdivision of $AG(r+1, 2)^*$, where $\eta = 2^r l$ and $r \geq 2$*

Small spectrum binary matroids

Theorem (Lemos, Reid, Wu 2011)

Let M be a 3-connected binary matroid with largest circuit size odd. Then $|spec(M)| \leq 2$ if and only if M is isomorphic to one of the following matroids.

(i) $U_{0,1}$ *or* $U_{2,3}$

(ii) S^*_{2n} *for some* $n \geq 2$

(iii) $B(r,2)^*$ *for some* $r \geq 2$

Bicircular matroids

The bicircular matroid of graph $G = (V, E)$, denoted by $B(G)$

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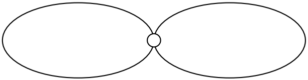
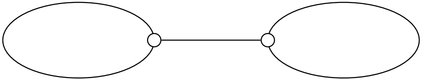
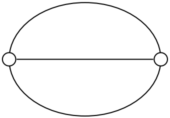
Bowtie, or tight handcuff



Barbell, or loose handcuff



Theta



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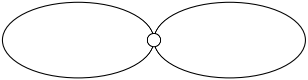

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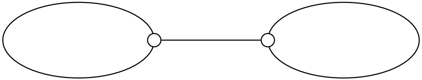

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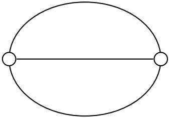


A bicycle is a connected subgraph containing exactly two cycles and no leaves.

A bicircular example

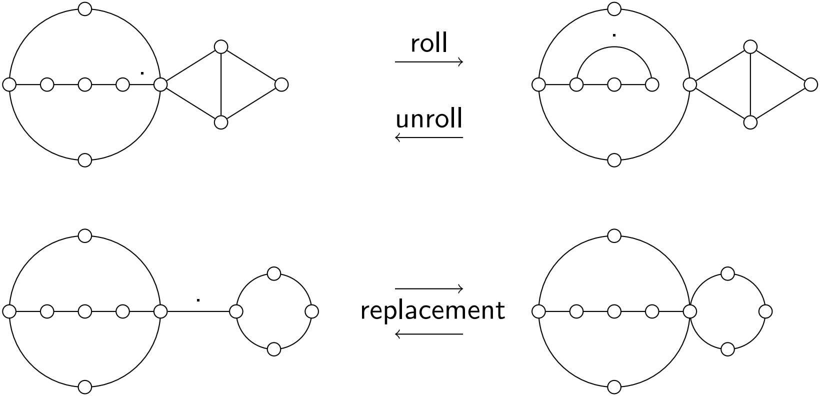
ground set: $E = \{a, b, c, d, e, f, g, h, i\}$

circuits:

$$\{a, b, d, e, f, g, h, i\} = E - \{c\}$$
$$\{a, b, c, d, f, g, h, i\} = E - \{e\}$$
$$\{a, b, c, d, e, g, h, i\} = E - \{f\}$$
$$\{c, d, e, f, g, h, i\} = E - \{a, b\}$$
$$\{a, b, c, e, f\} = E - \{d, g, h, i\}$$

Operations which preserve isomorphism of $B(M)$

(Coullard, del Greco, and Wagner 1991)



Small spectrum bicircular matroids

Bicircular matroid are generally not binary.

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Bicircular matroid are generally not binary.

Theorem (Lewis, McNulty, Neudauer, Reid, S 2013)

Let M be a connected bicircular matroid. For $\eta \geq 2$, spec$(M) = \{\eta\}$ if and only if M is isomorphic to one of the following matroids:

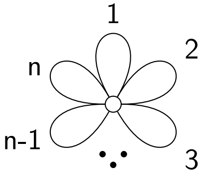
(i) *a k-subdivision of $U_{1,n}$ where $\eta = 2k$ and $n \geq 2$,*

(ii) *a k-subdivision of $U_{2,n}$ where $\eta = 3k$ and $n \geq 3$,*

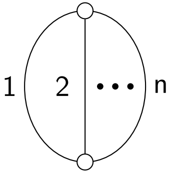
(iii) *a k-subdivision of $U_{3,5}$ or $U_{3,6}$ where $\eta = 4k$,*

(iv) *a k-subdivision of $U_{4,6}$ where $\eta = 5k$.*

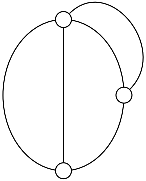
Small bicycle spectrum: $spec(M) = \{\eta\}$



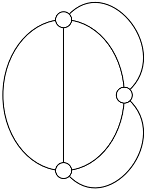
$U_{1,n}$



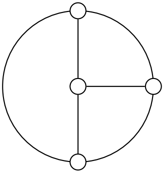
$U_{2,n}$



$U_{3,5}$



$U_{3,6}$



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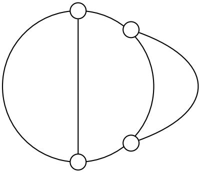

$U_{4,6}$

Small bicycle spectrum: $|spec(M)| = 2$

Theorem (Lewis, Reid, S)

Let $M = B(G)$ be a connected bicircular matroid where G is a subdivision of a 3-connected graph H. Then $|spec(M)| = 2$ if and only if H is one of the following graphs.

(i) *An (a, b)-subdivision of W_3 for distinct positive integers a, b.*

(ii) *A k-subdivision of W_4, $K_5 \backslash e$, K_5, $K_{3,3}$, $K_{3,4}$, or the prism P_6 for some $k \in \mathbb{Z}^+$.*
If H is isomorphic to W_4, $K_5 \backslash e$, or K_5, $spec(M) = \{5k, 6k\}$.
If H is isomorphic to $K_{3,3}$, $K_{3,4}$, or P_6, $spec(M) = \{6k, 7k\}$.

Small bicycle spectrum: $|spec(M)| = 2$

How does 3-connectivity matter?

Small bicycle spectrum: $|spec(M)| = 2$

Theorem (Dirac 1963)

A graph G is a subdivision of a simple 3-connected graph without two vertex-disjoint cycles if and only if G is a subdivision of one of the following graphs: a wheel graph, K_5, $K_5 \backslash e$, $K_{3,p}$, $K'_{3,p}$, $K''_{3,p}$, or $K'''_{3,p}$ for some $p \geq 3$.

Small bicycle spectrum: $|spec(M)| = 2$

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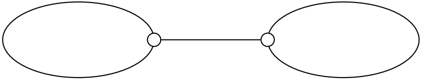
Two disjoint cycles . . .



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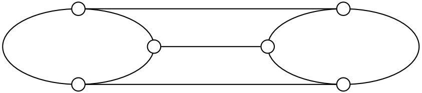
Two disjoint cycles . . . 3-connected



Small bicycle spectrum: $|spec(M)| = 2$

Theorem (Putnam, S)

Let $M = B(G)$ be a connected bicircular matroid where G is not the subdivision of a 3-connected graph. Then $|spec(M)| = 2$ if and only if G is a restricted *subdivision of one of the following graphs:*

(i) *A cycle with two or three balloons.*

(ii) *A theta with a balloon.*

(iii) *A theta barbell.*

(iv) *Two equally balanced thetas joined by two paths with the same endpoints.*

(v) *A theta barbell with a single balloon attached at either the center of the subdivided edge or at the branch point of a balanced theta.*

Small bicycle spectrum: $|spec(M)| = 3$

Theorem (Putnam, S)

Let $M = B(G)$ be a connected bicircular matroid where G is a subdivision of a 3-connected graph H. Then $|spec(M)| = 3$ if and only if G is one of the following graphs.

Small bicycle spectrum: $|spec(M)| = 3$

Theorem (Putnam, S)

Let $M = B(G)$ be a connected bicircular matroid where G is a subdivision of a 3-connected graph H. Then $|spec(M)| = 3$ if and only if G is one of the following graphs.

(i) *An (α, β, γ)-subdivision of W_3*

(ii) *An α-subdivision of W_5 or $K_{3,p}$ for $p \geq 4$*

(iii) *A $(\beta, 2\beta)$-subdivision of K_5 or $K_5 \setminus e$, with a matching being 2β subdivided*

(iv) *A $(\beta, 2\beta)$-subdivision of $K_{3,3}$ with exactly a single edge, a perfect matching, or a 4-cycle being 2β subdivided*

(v) *A $(\beta, 2\beta)$-subdivision of P_6 with exactly a matching or a 3-cycle being 2β subdivided*

(vi) *A restricted $(\beta, 2\beta)$-subdivision of W_4*

Large spectrum bicircular matroids

Which graphs have bicycles of many sizes?

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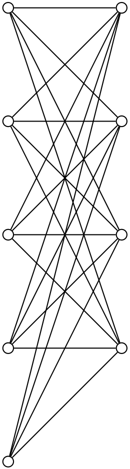
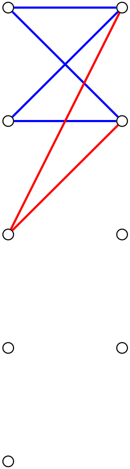


$K_{l,m}$ with $m \geq l \geq 2$ and $m \geq 3$ has bicycles of sizes

Large spectrum bicircular matroids

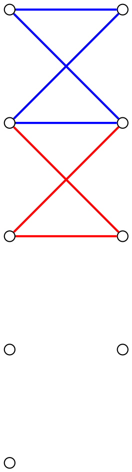
Which graphs have bicycles of many sizes?



$K_{l,m}$ with $m \geq l \geq 2$ and $m \geq 3$ has bicycles of sizes 6

Large spectrum bicircular matroids

Which graphs have bicycles of many sizes?



$K_{l,m}$ with $m \geq l \geq 2$ and $m \geq 3$ has bicycles of sizes 6, 7

Large spectrum bicircular matroids

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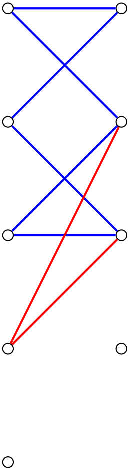
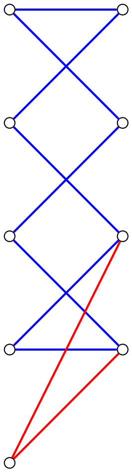


$K_{l,m}$ with $m \geq l \geq 2$ and $m \geq 3$ has bicycles of sizes 6, 7, 8, . . .

Large spectrum bicircular matroids

Which graphs have bicycles of many sizes?



$K_{l,m}$ with $m \geq l \geq 2$ and $m \geq 3$ has bicycles of sizes 6, 7, 8, $\ldots 2l + 2$

Consecutive cycle lengths

A question of Erdös, settled by Bondy and Vince (1998):
Every graph with minimum degree at least three contains two cycles whose lengths differ by one or two.

Consecutive cycle lengths

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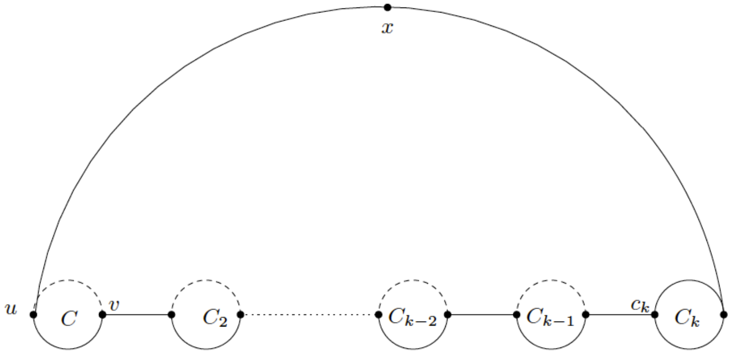
Every graph with minimum degree at least three contains two cycles whose lengths differ by one or two.

Theorem (Fan 2001)

Let xy be an edge in a 2-connected graph G. for an positive integer k, if every vertex other than x and y has degree at least $3k$, then xy is contained in $k+1$ cycles $C_0, C_1, \ldots C_k$ such that
$k+1 < |E(C_0)| < |E(C_1)| < \cdots < |E(C_k)|$,
$|E(C_i)| - |E(C_{i-1})| = 2$, $1 \leq i \leq k-1$, *and*
$1 \leq |E(C_k)| - |E(C_{k-1}| \leq 2$.

Using strings of cycles



A string of 2-defective cycles

Bicycles of consecutive sizes: strings of cycles

Adjusting this proof for bicycles, we get ...

Theorem (Putnam, S, Wu)

If G is a 2-connected graph with minimum degree at least $3k$ and G contains a non-separating induced odd cycle, then G contains $2(k-1)$ bicycles of consecutive sizes.

Bicycles of consecutive sizes: strings of cycles

Adjusting this proof for bicycles, we get . . .

Theorem (Putnam, S, Wu)

If G is a 2-connected graph with minimum degree at least $3k$ and G contains a non-separating induced odd cycle, then G contains $2(k-1)$ bicycles of consecutive sizes.

Theorem (Putnam, S, Wu)

Let x and y be two distinct vertices in a 2-connected graph G. If every vertex other than x and y has minimum degree at least $3k$,with $k \geq 2$, then G has $k-1$ bicycles,
$|E(C_1)| < |E(C_2)| < \cdots < |E(C_{k-1})|, \ \ |E(C_i)| - |E(C_{i-1})| \leq 2.$

Bicycles of consecutive sizes

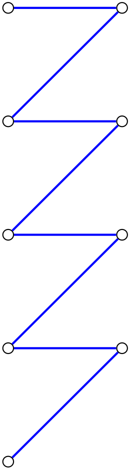
This is far from best possible.

Induced odd cycle produces all thetas, no barbells/handcuffs.

Produces multiple intervals of consecutive bicycle sizes, but no control on the gaps between intervals.

Bicycles of consecutive sizes: spanning trees

Adding any two edges to a spanning tree induces a bicycle.



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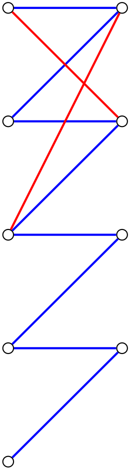
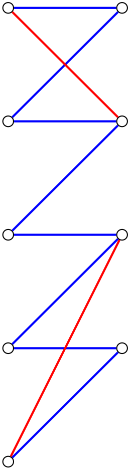


Bicycles of consecutive sizes: spanning trees

Adding any two edges to a spanning tree induces a bicycle.

If the longest path in a graph has length p, then the largest bicycle has size $p + 2$.



Bicycles of consecutive sizes

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Under what conditions will we have bicycles of all possible sizes?

Bicycles of consecutive sizes

Theorem

If G has a Hamilton path and minimum degree $k \geq 3$ then G has bicycles of k consecutive sizes, $n - k + 3 \leq |C| \leq n + 2$.

Bicycles of consecutive sizes

Theorem

If G has a Hamilton path and minimum degree $k \geq 3$ then G has bicycles of k consecutive sizes, $n - k + 3 \leq |C| \leq n + 2$.

Almost there

If G has minimum degree $k \geq 3$ and a maximal path of length p, then G has bicycles of $k - 1$ consecutive sizes, $p - k + 3 \leq |C| \leq p + 2$.

Utilizing a spanning tree

Choose a rooted depth-first-search spanning tree T rooted at one end v_0 of a maximal path $P = v_0, v_1, \ldots v_p$.

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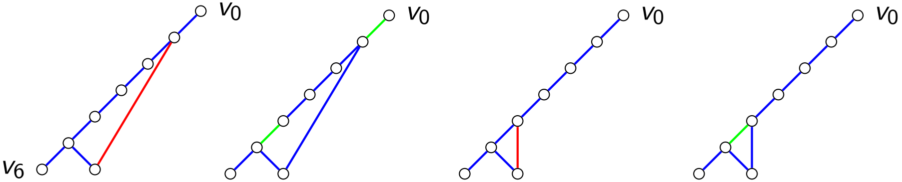
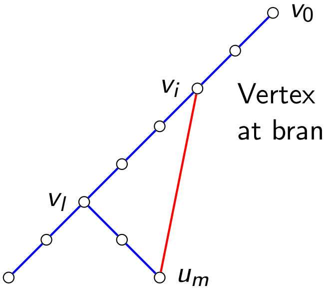
Choose a rooted depth-first-search spanning tree T rooted at one end v_0 of a maximal path $P = v_0, v_1, \ldots v_p$.
The neighbors (in G) of any vertex v lie on the path from v to v_0 in T.



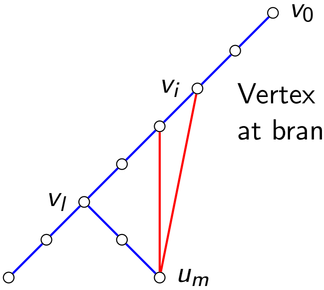
Utilizing a spanning tree



Vertex u_m is distance m from the path at branch v_l , with neighbor v_i

$$m < i < l - m$$

Utilizing a spanning tree



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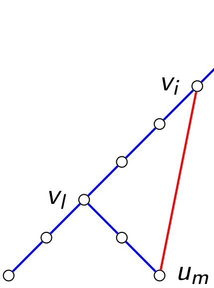
Utilizing a spanning tree



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Minimum degree $k \geq 3 \implies l \geq 2k$

Utilizing a spanning tree

Can we expect all bicycle sizes this way?

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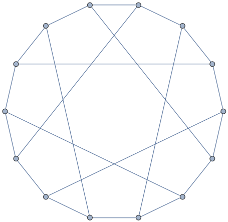

The Haewood graph has a full spectrum:
$\{9, 10, 11, 12, 13, 14, 15\}$

Utilizing a spanning tree

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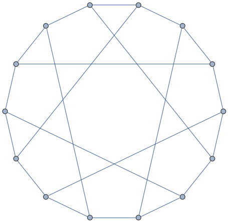

The Haewood graph has a full spectrum: $\{9, 10, 11, 12, 13, 14, 15\}$

Bicycles of length 10 require a different structure.


Some special matroids

The Uniform Matroid $U_{r,n} = (E, \mathcal{I})$ has $|E| = n$ and $\mathcal{I} = \{S \subseteq E : |S| \leq r\}$.
$PG(r, 2)$ is the binary projective geometry of rank $r + 1$.
$AG(r, 2)$ is the affine geometry of rank $r + 1$.
A matroid is binary provided it can be represented by (the columns of) a matrix with binary entries.
Every graphic matroid is binary. Bicircular matriods are generally not binary.