On bipartite $Q$-polynomial distance-regular graphs with $c_2 \leq 2$

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Outline

1. Basic definition and results from Algebraic graph theory
   (a.1) Distance-regular graphs, examples, hypercubes
   (a.2) Q-polynomial property of DRG
   (a.3) Result of Coughman, motivation

2. Bipartite Q-polynomial DRG with $D \geq 6$ and $c_2 \leq 2$
   Case $D \geq 6$ - Theorem 7.
   Case $D \geq 6$ - Proof of Theorem 7.

3. Equitable partitions when $c_2 \leq 2$
   The partition - part I
   The partition - part II

4. Case $D = 4$
   Theorem 35
Some notation before definition of DRG
A connected graph $\Gamma$ is called distance-regular (DRG) if there are numbers $a_i$, $b_i$, $c_i$ ($0 \leq i \leq D$) s.t. if $\partial(x, y) = h$ then

- $|\Gamma_1(y) \cap \Gamma_{h-1}(x)| = c_h$
- $|\Gamma_1(y) \cap \Gamma_h(x)| = a_h$
- $|\Gamma_1(y) \cap \Gamma_{h+1}(x)| = b_h$

Numbers $a_i$, $b_i$ and $c_i$ ($0 \leq i \leq D$) are called intersection numbers, and $\{b_0, b_1, ..., b_{D-1}; c_1, c_2, ..., c_D\}$ is intersection array.
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Distance-regular graphs - examples

- Line graph of Petersen’s graph.
Distance-regular graphs - examples

- Line graph of Petersen’s graph (diameter is three and intersection array is \(\{4, 2, 1; 1, 1, 4\}\))
Hamming graphs

- The Hamming graph $H(n, q)$ is the graph whose vertices are words (sequences or $n$-tuples) of length $n$ from an alphabet of size $q \geq 2$. Two vertices are considered adjacent if the words (or $n$-tuples) differ in exactly one term. We observe that $|V(H(n, q))| = q^n$.

- The Hamming graph $H(n, q)$ is distance-regular (with $a_i = i(q - 2)$ ($0 \leq i \leq n$), $b_i = (n - i)(q - 1)$ ($0 \leq i \leq n - 1$) and $c_i = i$ ($1 \leq i \leq n$)).
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Hamming graphs $H(3, 2)$.

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n-dimensional hypercubes (shortly n-cubes)

- Hamming graph $H(n, q)$ in which words of length $n$ are from an alphabet of size $q = 2$ are called $n$-dimensional hypercubes or shortly $n$-cubes.
4-dimensional hypercube (4-cubes)
More examples

- That comes from classical objects:
  - Hamming graphs,
  - Johnson graphs,
  - Grassmann graphs,
  - bilinear forms graphs,
  - sesquilinear forms graphs,
  - dual polar graphs (the vertices are the maximal totally isotropic subspaces on a vector space over a finite field with a fixed (non-degenerate) bilinear form)

- Some non-classical examples:
  - Doob graphs,
  - twisted Grassman graphs,

Distance-regular graphs give a way to study these classical objects from a combinatorial view point.
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- Distance-regular graphs give a way to study these classical objects from a combinatorial view point.
Distance-\(i\) matrix

- Let \(\text{Mat}_V(\mathbb{R})\) denote the algebra of matrices over \(\mathbb{R}\) with rows and columns indexed by \(V\).

- For \(0 \leq i \leq D\), let \(A_i\) denote the matrix in \(\text{Mat}_V(\mathbb{R})\) with \((y, z)\)-entry

\[
(A_i)_{yz} = \begin{cases} 
1 & \text{if } \partial(y, z) = i, \\
0 & \text{if } \partial(y, z) \neq i
\end{cases} \quad (y, z \in X).
\]

- We call \(A_i\) the \(i\)th distance-\(i\) matrix of \(\Gamma\).
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- We call $A_i$ the \textit{ith distance-i matrix} of $\Gamma$. 

Primitive idempotents

- We refer to $E_0, \ldots, E_D$ as the primitive idempotents of $\Gamma$.
- Primitive idempotents of $\Gamma$ represents the orthogonal projectors onto $\mathcal{E}_i = \ker(A - \theta_i I)$ (along $\text{im}(A - \theta_i I)$)
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Distance algebra

- If $\Gamma$ is regular (and $\Gamma$ is not distance-regular) we have:

\begin{align*}
A^0 &= A_0 = I \\
A^1 &= A_1 = A \\
J &= \Sigma A_i = H(A)
\end{align*}

- Adjacency algebra (ordinary "·" product), $A = \text{span}\{A^0, A^1, \ldots, A^d\} = \text{span}\{E_0, E_1, \ldots, E_d\}$

- Distance algebra (entry-wise "$\circ$" multiplication), $D = \text{span}\{A_0, A_1, \ldots, A_D\}$
The following statements are equivalent:

(i) $\Gamma$ is distance-regular,
(ii) $\mathcal{D}$ is an algebra with the ordinary product,
(iii) $\mathcal{A}$ is an algebra with the Hadamard product,
(iv) $\mathcal{A} = \mathcal{D}$. 

\[ A^0 = A_0 = I \]
\[ A^1 = A_1 = A \]
\[ J = \sum A_i = H(A) \]
Let $\Gamma$ denote any distance regular graph with diameter $D \geq 3$, and let $A$ denote the adjacency algebra for $\Gamma$. Let $E$ denote a primitive idempotent of $\Gamma$.

Since $A$ has a basis $A_0, A_1, \ldots, A_D$ of $0-1$ matrices, $A$ is closed under entry-wise matrix multiplication.

$\Gamma$ is said to be $Q$-polynomial with respect to $E = E_1$ whenever there exist an ordering $E_0, E_1, \ldots, E_D$ of the primitive idempotents such that for each $i$ ($0 \leq i \leq D$), the primitive idempotent $E_i$ is a polynomial of degree exactly $i$ in $E_1$, in the $\mathbb{R}$-algebra $(A, \circ)$, where $\circ$ denote entry-wise multiplication.

We say $\Gamma$ is $Q$-polynomial whenever $\Gamma$ is $Q$-polynomial with respect to at least one primitive idempotent.
Q-polynomial property

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- We say $\Gamma$ is $Q$-polynomial whenever $\Gamma$ is $Q$-polynomial with respect to at least one primitive idempotent.
Theorem (Caughman, 2004)

Let \( \Gamma \) denote a bipartite distance-regular graph with diameter \( D \geq 12 \). If \( \Gamma \) is \( Q \)-polynomial then \( \Gamma \) is either the ordinary \( 2D \)-cycle, or the \( D \)-dimensional hypercube, or the antipodal quotient of the \( 2D \)-dimensional hypercube, or the intersection numbers of \( \Gamma \) satisfy

\[
c_i = \frac{(q^i - 1)}{(q - 1)}, \quad b_i = \frac{(q^D - q^i)}{(q - 1)} \quad (0 \leq i \leq D)
\]

for some integer \( q \) at least 2.

- Note that if \( c_2 \leq 2 \), then the last of the above possibilities cannot occur.

- It is the aim of this presentation to further investigate graphs with \( D \leq 11 \) and \( c_2 \leq 2 \).
Result of Coughman, motivation

**Theorem (Caughman, 2004)**

Let $\Gamma$ denote a bipartite distance-regular graph with diameter $D \geq 12$. If $\Gamma$ is $Q$-polynomial then $\Gamma$ is either the ordinary $2D$-cycle, or the $D$-dimensional hypercube, or the antipodal quotient of the $2D$-dimensional hypercube, or the intersection numbers of $\Gamma$ satisfy $c_i = (q^i - 1)/(q - 1)$, $b_i = (q^D - q^i)/(q - 1)$ ($0 \leq i \leq D$) for some integer $q$ at least 2.

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Our main result is the following theorem.

**Theorem 1.**
Let $\Gamma$ denote a bipartite $Q$-polynomial distance-regular graph with diameter $D \geq 4$, valency $k \geq 3$, and intersection number $c_2 \leq 2$. Then one of the following holds:

(i) $\Gamma$ is the $D$-dimensional hypercube;  
(ii) $\Gamma$ is the antipodal quotient of the $2D$-dimensional hypercube;  
(iii) $\Gamma$ is a graph with $D = 5$ not listed above.
Theorem 7.

Let $\Gamma$ denote a $Q$-polynomial bipartite distance-regular graph with diameter $D \geq 6$, valency $k \geq 3$, and intersection numbers $b_i, c_i$.

In this section we show that if $c_2 \leq 2$, then $\Gamma$ is either the $D$-dimensional hypercube, or the antipodal quotient of the $2D$-dimensional hypercube.
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Idea for proof of Theorem 7.

- Assume that $\Gamma$ is not the $D$-dimensional hypercube or the antipodal quotient of the $2D$-dimensional hypercube.

- Then there exist scalars $s^*, q \in \mathbb{R}$ such that

$$c_i = \frac{h(q^i - 1)(1 - s^*q^{D+i+1})}{1 - s^*q^{2i+1}}, \quad b_i = \frac{h(q^D - q^i)(1 - s^*q^{i+1})}{1 - s^*q^{2i+1}}$$

$$h = \frac{1 - s^*q^3}{(q - 1)(1 - s^*q^{D+2})}$$
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Idea for proof of Theorem 7. (cont.)

- By [3, Lemma 4.1 and Lemma 5.1], scalars \( s^* \) and \( q \) satisfy

\[
q > 1, \quad \text{and} \quad -q^{-D-1} \leq s^* < q^{-2D-1}. \tag{1}
\]

- Assume first \( c_2 = 1 \). Abbreviate

\[
\alpha = 1 + q - q^2 - q^{D-1} + q^D + q^{D+1}
\]

and observe \( \alpha > 2 \). By Lemma 6(iii) we find

\[
s^* = \frac{\alpha \pm \sqrt{\alpha^2 - 4q^{D+1}}}{2q^{D+3}}.
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- Note that \( \alpha^2 - 4q^{D+1} \geq 0 \), and so we have

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- After some computation we show that

\[ s^* \geq \alpha - \frac{\alpha^2 - 4q^{D+1}}{2q^{D+3}} > q^{-2D-1}, \]

contradicting (1).

- Something similar we have also for \( c_2 = 2 \).
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Something similar we have also for \( c_2 = 2 \).
Definition of $D_{ij}$

Assume that $\Gamma = (X, R)$ is bipartite with diameter $D \geq 4$, valency $k \geq 3$ and intersection number $c_2 = 2$.

In this section we describe certain partition of the vertex set $X$.

Definition 8.

Let $\Gamma$ denote a bipartite distance-regular graph with diameter $D \geq 4$, valency $k \geq 3$ and intersection number $c_2 = 2$. Fix vertices $x, y \in X$ such that $\partial(x, y) = 2$. For all integers $i, j$ we define $D_{ij} = D_{ij}(x, y)$ by

$$D_{ij} = \{ w \in X \mid \partial(x, w) = i \text{ and } \partial(y, w) = j \}.$$  

We observe $D_{ij} = \emptyset$ unless $0 \leq i, j \leq D$. Moreover $|D_{ij}| = p_{ij}^2$ for $0 \leq i, j \leq D$. 


Definition of $D^i_j$

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We observe $D^i_j = \emptyset$ unless $0 \leq i, j \leq D$. Moreover $|D^i_j| = p_{ij}^2$ for $0 \leq i, j \leq D$. 
Definition of $D_j^i$ - examples

- 4-cube with sets $D_j^i$ ($b_0 = 4, b_1 = 3, b_2 = 2, b_3 = 1; c_1 = 1, c_2 = 2, c_3 = 3, c_4 = 4$).
Case $c_2 = 2$

- What if $c_2 = 2$?

**Definition 13.**

For $1 \leq i \leq D$ we define $A_i = A_i(x, y)$, $C_i = C_i(x, y)$, $B_i(z) = B_i(z)(x, y)$, $B_i(v) = B_i(v)(x, y)$ by

\[
A_i = \{ w \in D_i^i \mid \partial(w, z) = i + 1 \text{ and } \partial(w, v) = i + 1 \},
\]

\[
C_i = \{ w \in D_i^i \mid \partial(w, z) = i - 1 \text{ and } \partial(w, v) = i - 1 \},
\]

\[
B_i(z) = \{ w \in D_i^i \mid \partial(w, z) = i - 1 \text{ and } \partial(w, v) = i + 1 \},
\]

\[
B_i(v) = \{ w \in D_i^i \mid \partial(w, z) = i + 1 \text{ and } \partial(w, v) = i - 1 \}.
\]

We observe $D_i^i$ is a disjoint union of $A_i, B_i(z), B_i(v), C_i$. 

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**The partition - part I**

**The partition - part II**
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$$C_i = \{w \in \mathcal{D}_i^i | \partial(w, z) = i - 1 \text{ and } \partial(w, v) = i - 1\},$$

$$B_i(z) = \{w \in \mathcal{D}_i^i | \partial(w, z) = i - 1 \text{ and } \partial(w, v) = i + 1\},$$

$$B_i(v) = \{w \in \mathcal{D}_i^i | \partial(w, z) = i + 1 \text{ and } \partial(w, v) = i - 1\}.$$  

We observe $\mathcal{D}_i^i$ is a disjoint union of $A_i, B_i(z), B_i(v), C_i$.  

Case $c_2 = 2$ (cont.)

Partition of graph $\Gamma$, which involves $4(D - 1) + 2 \ell$ cells
Equitable partition

- We claim that the partition of $\mathcal{V}\Gamma$ into nonempty sets $D_{i+1}^{i-1}, D_{i-1}^{i+1}$ (1 ≤ $i$ ≤ $D - 1$), $A_i$ (2 ≤ $i$ ≤ $D - 1$), $B_i(z), B_i(v)$ (1 ≤ $i$ ≤ $D - 1$) and $C_i$ (3 ≤ $i$ ≤ $D$) is equitable.

- Main tool is "balanced set theorem".

Theorem (Terwilliger, 1995) (abridged version of theorem)

Let $\Gamma$ denote a distance-regular graph with diameter $D \geq 3$. Let $E$ denote a nontrivial primitive idempotent of $\Gamma$ and let $\{\theta^*_i\}_{i=0}^D$ denote the corresponding dual eigenvalue sequence. Then for all integers $h, i, j$ (1 ≤ $h$ ≤ $D$), (0 ≤ $i, j$ ≤ $D$) and for all $x, y \in \mathcal{X}$ such that $\partial(x, y) = h$,

$$\sum_{z \in \mathcal{X}} E\hat{z} - \sum_{z \in \mathcal{X}} E\hat{z} = p_{ij}^h \frac{\theta^*_i - \theta^*_j}{\theta^*_0 - \theta^*_h} (E\hat{x} - E\hat{y}).$$
Equitable partition

- We claim that the partition of $V\Gamma$ into nonempty sets $D_{i+1}^{i-1}, D_{i-1}^{i+1}$ ($1 \leq i \leq D - 1$), $A_i$ ($2 \leq i \leq D - 1$), $B_i(z), B_i(v)$ ($1 \leq i \leq D - 1$) and $C_i$ ($3 \leq i \leq D$) is equitable.
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Let $\Gamma$ denote a distance-regular graph with diameter $D \geq 3$. Let $E$ denote a nontrivial primitive idempotent of $\Gamma$ and let $\{\theta_i^*\}_{i=0}^D$ denote the corresponding dual eigenvalue sequence. Then for all integers $h, i, j$ ($1 \leq h \leq D$), ($0 \leq i, j \leq D$) and for all $x, y \in X$ such that $\partial(x, y) = h$,

$$\sum_{z \in X, \partial(x,z)=i} E\hat{z} - \sum_{z \in X, \partial(y,z)=j} E\hat{z} = p_{ij}^h \frac{\theta_i^* - \theta_j^*}{\theta_0^* - \theta_h^*} (E\hat{x} - E\hat{y}).$$
Equitable partition

- We claim that the partition of $V\Gamma$ into nonempty sets $D_{i+1}^{-1}, D_{i-1}^{i+1} (1 \leq i \leq D - 1)$, $A_i (2 \leq i \leq D - 1)$, $B_i(z), B_i(v) (1 \leq i \leq D - 1)$ and $C_i (3 \leq i \leq D)$ is equitable.
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Case $D = 4$

- In this section we consider $Q$-polynomial bipartite distance-regular graph $\Gamma$ with intersection number $c_2 \leq 2$, valency $k \geq 3$ and diameter $D = 4$.
- We show that $\Gamma$ is either the 4-dimensional hypercube, or the antipodal quotient of the 8-dimensional hypercube.
In this section we consider $Q$-polynomial bipartite distance-regular graph $\Gamma$ with intersection number $c_2 \leq 2$, valency $k \geq 3$ and diameter $D = 4$.

We show that $\Gamma$ is either the 4-dimensional hypercube, or the antipodal quotient of the 8-dimensional hypercube.
For the case $c_2 = 1$ we have the following result.

**Theorem (Miklavič, 2007)**

There does not exist a $Q$-polynomial bipartite distance-regular graph with diameter $D = 4$, valency $k \geq 3$ and intersection number $c_2 = 1$. 
For the case $c_2 = 1$ we have the following result.

**Theorem (Miklavič, 2007)**

There does not exist a $Q$-polynomial bipartite distance-regular graph with diameter $D = 4$, valency $k \geq 3$ and intersection number $c_2 = 1$. 
$c_2 = 2$ - Equitable partition
\( c_2 = 2 \) - ingredients

Let \( \Gamma \) denote a \( Q \)-polynomial bipartite distance-regular graph with diameter \( D = 4 \), valency \( k \geq 4 \) and intersection number \( c_2 = 2 \). Assume \( \Gamma \) is not the 4-dimensional hypercube or the antipodal quotient of the 8-dimensional hypercube.

- \( |\mathcal{A}_2| = (k - 2)(c_3 - 3)/2 \);
- \( c_3 \geq 4 \) if and only if \( \mathcal{A}_2 \neq \emptyset \);
- pick \( w \in \mathcal{A}_2 \) let \( \lambda \) denote number or neighbours of \( w \) in \( \mathcal{A}_3 \);
- \( \lambda = \frac{(k - 2)b_3(b_3 - 1)}{(k - 2)(k - 3) - 2b_3} \);
- \( (k - 2)(k - 3) - 2b_3 \) divides \( (k - 2)b_3(b_3 - 1) \).
Let $\Gamma$ denote a $Q$-polynomial bipartite distance-regular graph with diameter $D = 4$, valency $k \geq 4$ and intersection number $c_2 = 2$. Assume $\Gamma$ is not the 4-dimensional hypercube or the antipodal quotient of the 8-dimensional hypercube.

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- $\lambda = \frac{(k - 2)b_3(b_3 - 1)}{(k - 2)(k - 3) - 2b_3}$;
- $(k - 2)(k - 3) - 2b_3$ divides $(k - 2)b_3(b_3 - 1)$.
\[ c_2 = 2 - \text{ingredients} \]

- Let \( \Gamma \) denote a \( Q \)-polynomial bipartite distance-regular graph with diameter \( D = 4 \), valency \( k \geq 4 \) and intersection number \( c_2 = 2 \). Assume \( \Gamma \) is not the 4-dimensional hypercube or the antipodal quotient of the 8-dimensional hypercube.
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Basic definition and results from Algebraic graph theory
Bipartite $Q$-polynomial DRG with $D \geq 6$ and $c_2 \leq 2$
Equitable partitions when $c_2 \leq 2$
Case $D = 4$

$c_2 = 2$ - ingredients (cont.)

- Each vertex in $B_3(v)$ has exactly $\frac{(c_3 - 3)(b_3 - \lambda)}{b_3}$ neighbours in $A_2$.
- $(k - 2)(k - 3) - 2b_3$ divides $(k - 4)b_3(b_3 - 1)$
- $(k - 2)(k - 3) - 2b_3$ divides $2b_3(b_3 - 1)$;
- $(k - 2)(k - 3) = 2b_3^2$;
- $\lambda = (k - 2)/2$;
- $q = -(\sqrt{5} + 3)/2$;
- $s^* = 72\sqrt{5} - 161$. 
\( c_2 = 2 \) - ingredients (cont.)

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- \((k - 2)(k - 3) = 2b^2_3\);
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$c_2 = 2$ - ingredients (cont.)

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- $(k - 2)(k - 3) - 2b_3$ divides $2b_3(b_3 - 1)$;
- $(k - 2)(k - 3) = 2b_3^2$;
- $\lambda = (k - 2)/2$;
- $q = -(\sqrt{5} + 3)/2$;
- $s^* = 72\sqrt{5} - 161$. 
Theorem 35.
Let $\Gamma$ denote a $Q$-polynomial bipartite distance-regular graph with diameter $D = 4$, valency $k \geq 3$ and intersection number $c_2 = 2$. Then $\Gamma$ is either the 4-dimensional hypercube, or the antipodal quotient of the 8-dimensional hypercube.

Assume first that $c_3 \geq 4$. Then by Lemma 34 we have $q = -(\sqrt{5} + 3)/2$ and $s^* = 72\sqrt{5} - 161$. Lemma 6(iii) now implies $k = -6$, a contradiction. Therefore $c_3 = 3$. But now [4, Theorem 4.6] implies that $\Gamma$ is either the 4-dimensional hypercube, or the antipodal quotient of the 8-dimensional hypercube.
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Theorem 35


