Jacobians

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Rogla, Slovenia

27 June - 03 July 2015

REPUBLIKA SLOVENIJA
MINISTRSTVO ZA IZOBRAŽEVANJE, ZNANOST IN ŠPORT

Naložba v vašo prihodnost
OPERACIJO DELNO FINANCIRA EVROPSKA UNIJA
Evropski socialni sklad

Basic definitions

The notion of the Jacobian group of graph (also known as the Picard group, critical group, sandpile group, dollar group) was independently given by many authors (R. Cori and D. Rossin, M. Baker and S. Norine, N. L. Biggs, R. Bacher, P. de la Harpe and T. Nagnibeda). This is a very important algebraic invariant of a finite graph. In particular, the order of the Jacobain group coincides with the number of spanning trees for a graph. Following M. Baker and S. Norine we define the Jacobian group of a graph as follows.

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Let G be a finite, connected multigraph without loops. Let $V(G)$ and $E(G)$ be the sets of vertices and edges of G, respectively. Denote by $Div(G)$ a free Abelian group on $V(G)$. We refer to elements of $Div(G)$ as divisors on G. Each element $D \in Div(G)$ can be uniquely presented as $D = \sum_{x \in V(G)} D(x)(x)$, $D(x) \in \mathbb{Z}$. We define the *degree* of D by the formula $deg(D) = \sum_{x \in V(G)} D(x)$. Denote by $Div^0(G)$ the subgroup of $Div(G)$ consisting of divisors of degree zero.

Jacobians

Let f be a $\mathbb{Z}$-valued function on $V(G)$. We define the divisor of f by the formula

$$div(f) = \sum_{x \in V(G)} \sum_{xy \in E(G)} (f(x) - f(y))(x).$$

The divisor $div(f)$ can be naturally identified with the graph-theoretic Laplacian Δf of f. Divisors of the form $div(f)$, where f is a $\mathbb{Z}$-valued function on $V(G)$, are called *principal divisors*. Denote by $Prin(G)$ the group of principal divisors of G. It is easy to see that every principal divisor has a degree zero, so that $Prin(G)$ is a subgroup of $Div^0(G)$.
The *Jacobian group* (or *Jacobian*) of G is defined to be the quotient group

$$Jac(G) = Div^0(G)/Prin(G).$$

By making use of the Kirchhoff Matrix-Tree theorem one can show that $Jac(G)$ is a finite Abelian group of order $t(G)$, where $t(G)$ is the number of spanning trees of G. An arbitrary finite Abelian group is the Jacobian group of some graph.

Jacobians and flows

By results of M. Baker and S. Norine the Jacobian $Jac(G)$ is an Abelian group generated by flows $\omega(e), e \in \vec{E}$, whose defining relations are given by the two following Kirchhoff's laws.

(K_1) The flow through each vertex of G is equal to zero. It means that

$$\sum_{e \in \vec{E}, t(e)=x} \omega(e) = 0 \text{ for all } x \in V(G).$$

(K_2) The flow along each closed orientable walk W in G is equal to zero. That is

$$\sum_{e \in W} \omega(e) = 0.$$

The Smith normal form

Let $\mathcal{A}$ be a finite Abelian group generated by $x_1, x_2, \ldots, x_n$ and satisfying the system of relations

$$\sum_{j=1}^{n} a_{ij} x_j = 0, \; i = 1, \ldots, m,$$

where $A = \{a_{ij}\}$ is an integer $m \times n$ matrix. Set $d_j, \; j = 1, \ldots, r$, for the greatest common divisor of all $j \times j$ minors of A. Then,

$$\mathcal{A} \cong \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2/d_1} \oplus \mathbb{Z}_{d_3/d_2} \oplus \cdots \oplus \mathbb{Z}_{d_r/d_{r-1}}.$$

The Smith normal form

Two integral matrices A and B are equivalent (written $A \sim B$) if there are unimodular matrices P and Q such that $B = PAQ$. Equivalently, B is obtained from A by a sequence of elementary row and column operations: (1) the interchange of two rows or columns, (2) the multiplication of any row or column by -1, (3) the addition of any integer times of one row (resp. column) to another row (resp. column).

It is easy to see that $A \sim B$ implies that $\mathrm{coker(A)} \sim \mathrm{coker(B)}$. The *Smith normal form* is a diagonal canonical form for our equivalence relation: every $n \times n$ integral matrix A is equivalent to a unique diagonal matrix $\mathrm{diag}(s_1(A), \dots, s_n(A))$, where $s_i(A)$ divides $s_{i+1}(A)$ for $i = 1, 2, \dots, n-1$. The i-th diagonal entry of the Smith normal form of A is usually called the i-th invariant factor of A. We will use the fact that the values $s_i(A)$ can also be interpreted as follows: for each i, the product $s_1(A)s_2(A)\cdots s_i(A)$ is the greatest common divisor of all $i \times i$ minors of A.

Embedding graphs into Jacobians. Abel-Jacobi map.
For a fixed base point $x_0 \in V(G)$ we define the *Abel-Jacobi map* $S_{x_0} : G \to Jac(G)$ by the formula $S_{x_0}(x) = [(x) - (x_0)]$, where $[d]$ is the equivalence class of divisor d. The continuous version of the following theorem is well known in complex analysis.

Theorem (M. Baker and S. Norine, 2009)

If graph G is 2-edge connected (=bridgeless) then S_{x_0} is an imbedding.

Jacobians and Laplacians

Consider the Laplacian matrix $L(G)$ as a homomorphism $\mathbb{Z}^{|V|} \to \mathbb{Z}^{|V|}$, where $|V| = |V(G)|$ is the number of vertices of G. Then $\mathrm{coker}(L(G)) = \mathbb{Z}^{|V|}/\mathrm{im(L(G))}$ is an abelian group. Let

$$\mathrm{coker}(L(G)) \cong \mathbb{Z}_{t_1} \oplus \mathbb{Z}_{t_2} \oplus \cdots \oplus \mathbb{Z}_{t_{|V|}},$$

be its Smith normal form satisfying $t_i \big| t_{i+1}$, $(1 \leq i \leq |V|)$. If graph G is connected then the groups $\mathbb{Z}_{t_1}, \mathbb{Z}_{t_1}, \ldots \mathbb{Z}_{t_{|V|-1}}$ are finite and $\mathbb{Z}_{t_{|V|}} = \mathbb{Z}$. In this case,

$$Jac(G) = \mathbb{Z}_{t_1} \oplus \mathbb{Z}_{t_2} \oplus \cdots \oplus \mathbb{Z}_{t_{|V|-1}}$$

is the Jacobian group of the graph G.

Jacobians and harmonic maps

Let $\varphi : G \to G'$ be a harmonic morphism. We define the *push-forward* homomorphism $\varphi_* : \mathrm{Div}(G) \to \mathrm{Div}(G')$ by

$$\varphi_*(D) = \sum_{x \in V(G)} D(x)\varphi(x).$$

Similarly, we define the *pullback* homomorphism $\varphi^* : \mathrm{Div}(G') \to \mathrm{Div}(G)$ by

$$\varphi^*(D') = \sum_{x \in V(G')} \sum_{x \in V(G),\, \varphi(x) = y} m_\varphi(x) D'(y) x = \sum_{x \in V(G)} m_\varphi(x) D'(\varphi(x)) x.$$

Jacobians and harmonic maps

We note that if $\varphi : G \to G'$ is a harmonic morphism and $D' \in \mathrm{Div}(G')$, then

$$\deg(\varphi^*(D')) = \deg(\varphi) \cdot \deg(D').$$

Since, $\deg(\varphi) = \sum_{x \in V(G),\, \varphi(x)=y} m_{\varphi}(x)$ we also have the following simple formula:

Lemma 1.

Let $\varphi : G \to G'$ be a harmonic morphism, and let $D' \in \mathrm{Div}(G')$. Then

$$\varphi_*(\varphi^*(D')) = \deg(\varphi) D'.$$

Jacobians

Jacobians and harmonic maps Suppose $\varphi : G \to G'$ is a harmonic morphism and that $f : V(G) \to A$ and $f' : V(G') \to A$ are functions, where A is an abelian group. We define $\varphi_* f : V(G') \to A$ by

$$\varphi_* f(y) := \deg(\varphi) = \sum_{x \in V(G),\, \varphi(x)=y} m_\varphi(x) f(x)$$

and $\varphi^* f' : V(G) \to A$ by $\varphi^* f' := f' \circ \varphi$.

Lemma 2.

Let $\varphi : G \to G'$ be a harmonic morphism, and $f : V(G) \to Z$ and $f' : V(G') \to Z$ are functions. Then

$$\varphi_*(\mathrm{Div}(f)) = \mathrm{Div}(\varphi_* f),\ \varphi_*(\mathrm{Prin}(G)) \subseteq \mathrm{Prin}(G')$$

and

$$\varphi^*(\mathrm{Div}(f)) = \mathrm{Div}(\varphi^* f),\ \varphi^*(\mathrm{Prin}(G')) \subseteq \mathrm{Prin}(G).$$

Jacobians and harmonic maps As a consequence of Lemma 2, we see that φ induces group homomorphisms (which we continue to denote by φ_*, φ^*)

$$\varphi_* : \mathrm{Jac}(G) \to \mathrm{Jac}(G'),\ \varphi^* : \mathrm{Jac}(G') \to \mathrm{Jac}(G).$$

It is straightforward to check that if $\psi : G \to G'$ and $\varphi : G' \to G''$ are harmonic morphisms and $D \in \mathrm{Div}(G)$, $D'' \in \mathrm{Div}(G'')$, then $\varphi \circ \psi : G \to G''$ is harmonic, and we have

$$(\varphi \circ \psi)_*(D) = \varphi_*(\psi_*(D))$$

and

$$(\varphi \circ \psi)^*(D'') = \psi^*(\varphi^*(D'')).$$

Therefore we obtain two different functors from the category of graphs to the category of abelian groups:
a *covariant "Albanese"* functor $(G \to \mathrm{Jac}(G),\ \varphi \to \varphi_*)$ and
a *contravariant "Picard"* functor $(G \to \mathrm{Jac}(G),\ \varphi \to \varphi^*)$.

Jacobians and harmonic maps

As a result we the following two important theorems.

Theorem

Let $\varphi : G \to G'$ be a nonconstant harmonic morphism. Then $\varphi_* : \mathrm{Jac}(G) \to \mathrm{Jac}(G')$ is surjective.

Theorem

Let $\varphi : G \to G'$ be a nonconstant harmonic morphism. Then $\varphi^* : \mathrm{Jac}(G') \to \mathrm{Jac}(G)$ is injective.

Jacobians and harmonic maps

Let $t(G) = |\mathrm{Jac}(G)|$ denote the number of spanning trees in a graph G. From each of the two above theorems we immediately deduce the following corollary, a special case of which is due to K. A. Berman and D. Treumann.

Corollary

If there exists a nonconstant harmonic morphism from G to G', then $t(G')$ divides $t(G)$.

Exercises

Exercise 5.1.
Let G be a tree. Show that $Jac(G) = 0$.

Exercise 5.2.
Let C_n be a cyclic graph on n vertices. Show that $Jac(C_n) = \mathbb{Z}_n$.

Exercise 5.3.
Let K_n be the complete graph on n vertices. Prove that $Jac(K_n) = \mathbb{Z}_n^{n-2}$.

Exercise 5.4.
Let $K_{m,n}$ be the complete bipartite graph. Prove that
$Jac(K_{m,n}) = \mathbb{Z}_m^{n-2} \oplus \mathbb{Z}_n^{m-2} \oplus \mathbb{Z}_{mn}$.
(See: D.J. Lorenzini, A finite group attached to the Laplacian of a graph. Discrete Math. 91 (1991), 277–282.)

Exercise 5.5.
Let X be a finite connected graph. Denote by $\overline{X}$ the graph obtained from X by collapsing all bridges of X to vertices. Prove $Jac(X) = Jac(\overline{X})$.

Exercise 5.6.
Let X_1 and X_2 be connected graphs sharing a common vertex. Show that $Jac(X_1 + X_2) = Jac(X_1) \oplus Jac(X_2)$.

Exercise 5.7.
Let $A = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \ldots \oplus \mathbb{Z}_{n_r}$ be a finite Abelian group and $X = C_{n_1} + C_{n_2} + \ldots + C_{n_r}$. Show that $Jac(X) \cong A$.

Exercise 5.8.
Let e be an edge of graph X such that $X \setminus e = X_1 \cup X_2$ is a disjoint union of two connected graphs X_1 and X_2. Prove that
$Jac(X) = Jac(X_1) \oplus Jac(X_2)$.

Exercise 5.9.

Describe all bridgeless graphs X with $Jac(X) = G$, where

(i) $G = \mathbb{Z}_2$,

(ii) $G = \mathbb{Z}_3$,

(iii) $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2$,

(iv) $G = \mathbb{Z}_4$,

(v) $G = \mathbb{Z}_5$.

Exercise 5.10.

Let X and X^* be dual planar graphs. Prove that $Jac(X) = Jac(X^*)$.

Exercise 5.11. Find the Jacobian $J(W_n)$ of the wheel graph on $n+1$ vertices.

Answer: $Jac(W_n) = \mathbb{Z}_{\ell_n} \oplus \mathbb{Z}_{\ell_n}$, if n is odd and $Jac(W_n) = \mathbb{Z}_{f_n} \oplus \mathbb{Z}_{5f_n}$, if n is even. Here ℓ_j is j-th Lukas number and f_k is k-th Fibonacci number.
$\ell_1 = 1,\ \ell_2 = 3,\ \ell_{k+2} = \ell_{k+1} + \ell_k,\ k \geq 1.$
$f_1 = 1,\ f_2 = 1,\ f_{k+2} = f_{k+1} + f_k,\ k \geq 1.$
In particular, $Jac(W_2) = \mathbb{Z}_1 \oplus \mathbb{Z}_5, Jac(W_3) = \mathbb{Z}_4 \oplus \mathbb{Z}_4,$
$Jac(W_4) = \mathbb{Z}_3 \oplus \mathbb{Z}_{15}, Jac(W_5) = \mathbb{Z}_{11} \oplus \mathbb{Z}_{11}, Jac(W_6) = \mathbb{Z}_8 \oplus \mathbb{Z}_{40}.$