# Harmonic automorphisms of graphs

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# Introduction

The theory of Riemann surfaces was founded in classical works by B. Riemann and A. Hurwitz. We note that originally Riemann surface was defined as a branched covering over the sphere. Over the last decade, a few discrete versions of the theory of Riemann surfaces were created.

1. Bacher, R., P. de la Harpe, and Nagnibeda,T., 1997
2. H. Urakawa, H, 2000
3. Baker, M., Norine, S., 2009
4. Caporaso, L., 2011
5. Corry, S., 2013

In these theories, the role of Riemann surfaces is played by graphs, while the branched coverings are replaces by harmonic morphisms of graphs.

# Introduction

*Dictionary*

1. Riemann surface $\Longleftrightarrow$ Finite connected graph
2. Riemann surface with border $\Longleftrightarrow$ Finite connected graph with semi-edges
3. Holomorphic map (branched covering) $\Longleftrightarrow$ Harmonic map (quasi-covering)
4. The sphere $\Longleftrightarrow$ Tree
5. Torus (= one "hole" surface) $\Longleftrightarrow$ Flower (= one cycle graph)
6. Genus ($\sharp$ of "holes") $\Longleftrightarrow$ Genus ($\sharp$ of independent loops)
7. Conformal automorphism $\Longleftrightarrow$ Automorphism acting free on darts

# Harmonic group action on graphs

Let $X$ be a finite connected graph.

**Definition**

Suppose that $G < \mathrm{Aut}(X)$ is a group of automorphisms of a graph $X$. Then $G$ acts *harmonically* on $X$ if it acts freely (that is without fixed points) on the set of darts $D(X)$ of graph $X$.

We have the following observation.

**Observation (Scott Corry, Roman Nedela)**

*Let group $G$ act on a graph $X$ harmonically. Then the quotient map $X \to X/G$ is a harmonic morphism.*

# Harmonic group action on graphs

Recall some classical results for Riemann surface theory. For each $g \geq 2$ define

$$N(g) := \max\{|\mathrm{Aut}(S_g)| : \ S_g \text{ is a compact Riemann surface of genus } g\}.$$

Then

$$8(g+1) \leq N(g) \leq 84(g-1),$$

and these bounds are sharp in the sense that both the upper and lower bound are attained for infinitely many values of $g$. The upper bound was found by Hurwitz (1893). The lower bound was independently obtained by R. Accola (1968) and C. Maclachlan (1969).

# Harmonic group action on graphs

Denote by $M(g)$ maximum size of a harmonic group action on any graph of genus $g \geq 2$.

**Theorem (Scott Corry, 2011)**

*For $g \geq 2$ we have*

$$4(g-1) \leq N(g) \leq 6(g-1).$$

*The upper and lower bound are attained for infinitely many values of $g$.*

Recent paper by Scott Corry (2013) states that maximal graph groups $G$ with $|G| = 6(g-1)$ are exactly the finite quotients of the modular group $\Gamma =< x, y | x^2 = y^3 = 1 >$ of size at least 6.

# Oikawa theorem

In 1956 Kotaro Oikawa proved the following theorem.

**Theorem (Oikawa, 1956)**

*Let $S_g$ be a closed Riemann surface of genus $g$ and $A$ is a finite subset of $S_g$ consisting of $|A| \geq 1$ elements. Suppose that $2g - 2 + |A| > 0$ and $G$ is a group of conformal automorphisms of $S_g$ leaving the set $A$ invariant. Then*

$$|G| \leq 12(g - 1) + 6|A|.$$

In the next section we find a discrete version of the Oikawa's. Again, the key point of the proof is the Riemann-Hurwitz relation.

# Oikawa's theorem for graphs

Our result for graphs is the following theorem.

**Theorem 4 (R. Nedela, A. Mednykh, 2013)**

*Let $X$ be a graph of genus $g$ and $A$ is a subset of vertices of $X$ consisting of $|A| \geq 1$ elements. Suppose that $g - 1 + |A| > 0$ and $G$ is a finite group acting on $X$ harmonically and leaving the set $A$ invariant. Then*

$$|G| \leq 2(g-1) + 2|A|.$$

The upper bound is sharp and is attained for arbitrary large values of $g$ and $|A|$. So, at least infinitely many often.

# Proof of Oikawa's theorem

Let $G$ be a finite group acting pure harmonically on a graph $X$. For every $\tilde{v} \in V(X)$ denote by $G_{\tilde{v}}$ the stabiliser of $\tilde{v}$ in the group $G$ and by $|G_{\tilde{v}}|$ the order of the stabiliser. Then to each vertex $v \in V(X/G)$ we prescribe the number $m_v = |G_{\tilde{v}}|$, where $\tilde{v} \in \varphi^{-1}(v)$. Since $G$ acts transitively on each fibre of $\varphi$ the numbers $m_v$ are defined correctly.

The following version of the Riemann-Hurwitz formula can be found in [Baker-Norine] and [Mednykh]. The *genus* of a graph $X$ is defined as the rank of the first homology group.

## Proposition

*Let $G$ be a finite group acting harmonically on a graph $X$ of genus $g$. Denote by $\gamma$ genus of the factor graph $X/G$. Then*

$$g - 1 = |G|(\gamma - 1 + \sum_{v \in V(X/G)} (1 - \frac{1}{m_v})),$$

# Proof of Oikawa's theorem

Preliminary, we establish the following result ("Anti-Hurwitz"Lemma).

## Lemma

*Let $G$ be a finite group acting pure harmonically on a graph $X$ of genus $g$. Denote by $I \subset V(X)$ a $G$-invariant subset of vertices of $X$. Set $s = |I|$ and $p = |I/G|$. Then*

$$s = |G|(p - \sum_{v \in V(I/G)} (1 - \frac{1}{m_v})),$$

*where the numbers $m_v$ are the same as above.*

**Proof.** Since $p = \sum_{v \in V(I/G)} 1$ we have

$$\begin{aligned} |G|(p - \sum_{v \in V(I/G)} (1 - \frac{1}{m_v})) &= |G|(\sum_{v \in V(I/G)} 1 - \sum_{v \in V(I/G)} (1 - \frac{1}{m_v})) \\ &= |G| \sum_{v \in V(I/G)} \frac{1}{m_v} = \sum_{v \in V(I/G)} \frac{|G|}{m_v} \\ &= \sum_{v \in V(I/G)} |\varphi^{-1}(v)| = |I| = s. \end{aligned} \qquad (1)$$

# Proof of Oikawa's theorem

Now we are able to prove the following proposition.

## Proposition

*Let $G$ be a finite group acting pure harmonically on a graph $X$ of genus $g$. Suppose that $I \subset V(X)$ a $G$-invariant subset of vertices of $X$ and set $s = |I|$ and $p = |I/G|$. Denote by $\gamma$ genus of the factor graph $X/G$. Then*

$$g - 1 + s = |G|(\gamma - 1 + \sum_{v \in V(X/G) - I/G} (1 - \frac{1}{m_v}) + p),$$

*where the numbers $m_v$ are the same as above.*

**Proof.** The desired result is the sum of two formulas. The first one is given by the previous Proposition and the second by the Anti-Hurwitz Lemma.

**Proof of of Oikawa's theorem.** We will use the same notations $s = |I|$, $p = |I/G|$ and $m_v$, $v \in V(X/G)$ as above. Since we are going to apply Proposition we are interested only in $v \in V(X/G) - I/G$ with $m_v > 1$. Suppose that there are exactly $r \geq 0$ such vertices. Namely, $v_1, v_2, \ldots, v_r$. We set $m_i = m_{v_i}$, $i = 1, \ldots, r$. Abusing the language we will refer to $v_i$ as branched points of order $m_i$. Then by the above Proposition we have

$$g - 1 + s = |G|(\gamma - 1 + \sum_{i=1}^{r}(1 - \frac{1}{m_i}) + p). \tag{2}$$

# Proof of Oikawa's theorem

We set

$$A = \gamma - 1 + \sum_{i=1}^{r}(1 - \frac{1}{m_i}) + p.$$

Then $|G| = \frac{g-1+s}{A}$. By the assumption of theorem we get $A > 0$. Now, to find the upper bound for $|G|$ we have to find $\min A$ under condition $A > 0$. Now we consider two cases.

1° $\gamma \geq 1$, $p \geq 1$. Then $A \geq 0 + 0 + p \geq 1$ and the minimum value $A = 1$ is attained for $\gamma = 1$, $p = 1$, $r = 0$. In this case $X$ is a regular $G$-covering of a graph of genus $\gamma = 1$ branched over one point.

2° $\gamma = 0$, $p \geq 1$. Then $A \geq \sum\limits_{i=1}^{r}(1 - \frac{1}{m_i}) \geq \frac{1}{2}$. The minimum $A = \frac{1}{2}$ is attained for $\gamma = 0$, $p = 1$, $r = 1$ and $m_1 = 2$. In this case $X$ is a regular $G$-covering of a tree branched over $p + r = 2$ points with branch orders 2 and $s$.

# Proof of Oikawa's theorem

Finally, $A \geq \frac{1}{2}$. As a result we obtain

$$|G| = \frac{g-1+s}{A} \leq 2(g-1+s) = 2(g-1)+2s.$$

# Two Arakawa's theorems

Now our aim is to find discrete versions of two Arakawa's theorems (2000).

The first one states that if $G$ be a finite group of automorphisms of a compact Riemann surface $X$ of genus $g \geq 2$ and $A$ and $B$ are two disjoint $G$-invariant subsets of $X$ of the orders $|A| \geq |B| \geq 1$ then $|G| \leq 8(g-1) + |A| + 4|B|$.

The second theorem asserts that if $A, B$ and $C$ are three disjoint the $G$-invariant subsets of $X$ with $|A| \geq |B| \geq |C| \geq 1$ then $|G| \leq 2(g-1) + |A| + |B| + |C|$.

# Two Arakawa's theorems

We present a discrete version of the first Arakawa's theorem by the following theorem.

**Theorem 5 (R. Nedela, A. Mednykh and I. Mednykh 2013)**

*Let $X$ be a graph of genus $g \geq 2$ and $A$ and $B$ are two disjoint subsets of vertices of $X$ of the orders $|A| \geq |B| \geq 1$. Suppose that $G$ is a finite group acting harmonically on $X$ and leaving the sets $A$ and $B$ invariant. Then*

$$|G| \leq \frac{3(g-1)+|A|+3|B|}{2}.$$

Again, the upper bound is sharp and is attained for arbitrary large values of $g$ and $s$.

# Two Arakawa's theorems

A discrete version of the second Arakawa's theorem is given by the following theorem.

**Theorem 6 (R. Nedela, A. Mednykh and I. Mednykh, 2013)**

*Let $X$ be a graph of genus $g \geq 2$ and $A, B$ and $C$ are three disjoint subsets of vertices of $X$ of the orders $|A| \geq |B| \geq |C| \geq 1$. Suppose that $G$ is a finite group acting harmonically on $X$ and leaving the sets $A, B$ and $C$ invariant. Then*

$$|G| \leq \frac{g - 1 + |A| + |B| + |C|}{2}.$$

As in the two previous theorems, the upper bound is sharp and is attained for arbitrary large values of $g$ and $s$.

# Proof of the second Arakawa's theorems

The proof of this theorem is based on the following considerations. We set $I = A \cup B \cup C, s = |I| = |A| + |B| + |C|$, and $p = |I/G|$. Since $A, B$, and $C$ are pairwise disjoint, it follows that $p \geq 3$. The generalized Riemann-Hurwitz formula yields

$$g - 1 + s = |G|(\gamma - 1 + \sum_{i=1}^{r}(1 - \frac{1}{m_i}) + p) \geq |G|(0 - 1 + 0 + 3) = 2|G|.$$

Since $s = |A| + |B| + |C|$, the latter implies

$$g - 1 + |A| + |B| + |C| \geq 2|G|.$$

# Wiman's theorem

Klein's quartic curve, $x^3y + y^3z + z^3x = 0$, admits the group $\mathrm{PSL}_2(7)$ as its full group of conformal automorphisms. It is characterised as the curve of smallest genus realising the upper bound $84(g-1)$ on the order of a group of conformal automorphisms of a curve of genus $g > 1$, given by A. Hurwitz in 1893. Around the same time, A. Wiman (1895) characterised the curves $w^2 = z^{2g+1} - 1$ and $w^2 = z(z^{2g} - 1), g > 1$, as the unique curves of genus $g$ admitting cyclic automorphism groups of the largest and the second largest possible order ($4g + 2$ and $4g$, respectively). The modern proof of these and similar results is contained in the paper by K. Nakagawa (1984).

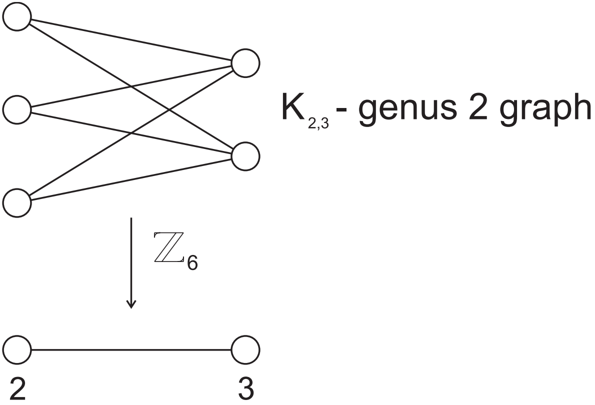
# Wiman's theorem

The aim of the present section is to find a discrete version of the Wiman theorem.

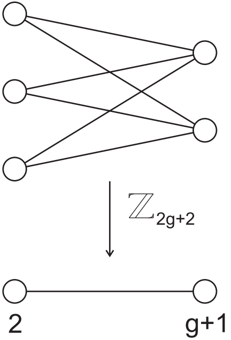
**Theorem 7 (A. Mednykh and I. Mednykh, 2013)**

*Let $X$ be a graph of genus $g \geq 2$ and $\mathbb{Z}_N$ is a cyclic group acting harmonically on $X$. Then $N \leq 2g + 2$. The upper bound $N = 2g + 2$ is attained for any even $g$. In this case, the signature of orbifold $X/\mathbb{Z}_N$ is $(0; 2, g + 1)$, that is, $X/\mathbb{Z}_N$ is a tree with two branch points of order 2 and $g + 1$, respectively.*

# Wiman's theorem



# Wiman's theorem




$K_{2,g+1}$ - genus g graph

g - even

# Wiman's theorem

The second and the third largest cyclic groups are given by

## Theorem 8 (A. Mednykh and I. Mednykh, 2013)

*Let $X$ be a graph of genus $g \geq 2$ and $\mathbb{Z}_N$ is a cyclic group acting harmonically on $X$. Let $N < 2g + 2$ then $N \leq 2g$. The upper bound $N = 2g$ is attained only in the following cases:*

(i) $N = 2g$ *and* $X/\mathbb{Z}_N$ *is an orbifold of the signature* $(0; 2, 2g), g \geq 2$;

(ii) $N = 12$ *and* $X/\mathbb{Z}_N$ *is an orbifold of the signature* $(0; 3, 4)$, $g = 6$.

*Also, if $N < 2g$ then $N \leq 2g - 1$. The upper bound $N = 2g - 1$ is attained only in two cases:*

(iii) $N = 3$ *and* $X/\mathbb{Z}_N$ *is an orbifold of the signature* $(0; 3, 3)$, $g = 2$;

(iv) $N = 15$ *and* $X/\mathbb{Z}_N$ *is an orbifold of the signature* $(0; 3, 5)$, $g = 8$.

# Wiman's theorem

The proofs of the last two theorems are based on the discrete version of the Riemann-Hurwitz formula, Bass-Serre uniformization theory for graphs of groups and the following elementary lemma.

**Lemma**

*Let $\Gamma = \mathbb{Z}^{\star\gamma} \star \mathbb{Z}_{m_1} \star \mathbb{Z}_{m_2} \star \ldots \star \mathbb{Z}_{m_r}$ be a free product of cyclic groups, and let $M = LCM(m_1, \ldots, m_r)$. Then there exists a homomorphism of the group $\Gamma$ to the cyclic group $\mathbb{Z}_N$ preserving the orders of elements if and only if $M$ divides $N$ and, in the case $\gamma = 0$, the equality $M = N$ holds.*