# Coverings of graphs and uniformisation theory

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**Graph coverings and covering groups**

Let $X$ and $Y$ be connected graphs. A surjective morphism $\varphi : X \to Y$ is called a *(graph) covering* if for any vertex $x \in V(X)$ the restriction $\varphi|_{\mathrm{St}_X(x)} : \mathrm{St}_X(x) \to \mathrm{St}_Y(\varphi(x))$ is an isomorphism. The coverings $\varphi : X \to Y$ and $\varphi' : X' \to Y$ are said to be *equivalent* if there is an isomorphism $h : X \to X'$ such that $\varphi = \varphi' \circ h$.
A *covering group* of $\varphi$ is defined as

$$\mathrm{Cov}(\varphi) = \{h \in \mathrm{Aut}(X) : \varphi = \varphi \circ h\}.$$

The covering $\varphi$ is called *regular* if $\mathrm{Cov}(\varphi)$ act transitively on each fibre of $\varphi$ and *irregular* otherwise. If $\varphi : X \to Y$ is a regular covering then $Y \cong X/\mathrm{Cov}(\varphi)$. A finite sheeted covering $\varphi : X \to Y$ is regular if and only if the order of covering group $|Cov(\varphi)|$ coincides with the number of sheets of the covering.

**Graph coverings and voltage assignments**

Permutation voltage assignments were introduced by J. L. Gross and T. W. Tucker. Let $X$ be a finite connected graph, possibly including multiple edges or loops. It is *directed* if each edge (even a loop) is provided by the two possible directions. Let $D(X)$ be the set of the directed edges of $X$ (also known as *darts, arcs* and so on in the literature). A *permutation voltage assignment* of $X$ with voltages in the symmetric group $\mathbb{S}_n$ of degree $n$ is a function $\phi : D(X) \to \mathbb{S}_n$ such that inverse edges have inverse assignments. The pair $(D(X), \phi)$ is called a permutation voltage graph.

**Graph coverings and voltage assignments**

The *(permutation) derived graph* $X^{\phi}$ derived from a permutation voltage assignment $\phi$ is defined as follows: $V(X^{\phi}) = V(X) \times \{1, \cdots, n\}$, and $((u,j), (v,k)) \in D(X^{\phi})$ if and only if $(u,v) \in D(X)$ and $k = \phi(u,v)(j)$. The natural projection $\pi : X^{\phi} \to X$ that is a function from $V(X^{\phi})$ onto $V(X)$ which erases the second coordinates gives a *graph covering*. J. L. Gross and T. W. Tucker showed that every covering of a given graph arises from some permutation voltage assignment in a symmetric group. Moreover, such a covering is connected if and only if $\phi(D(X))$ is a transitive subgroup in $\mathbb{S}_n$.

**Regular coverings and ordinary voltage assignments**

Ordinary voltage assignments were introduced by J. L. Gross. Let $G$ be a finite group. Then a mapping $\omega : D(X) \to G$ is called an *ordinary voltage assignment* if $\omega(v, u) = \omega(u, v)^{-1}$ for each $(u, v) \in D(X)$. The *(ordinary) derived graph* $X^{\omega}$ derived from an ordinary voltage assignment $\omega$ is defined as follows: $V(X^{\omega}) = V(X) \times G$, and $((u, j), (v, k)) \in D(X^{\omega})$ if and only if $(u, v) \in D(X)$ and $k = \omega(u, v)j$. Consider the natural projection $\pi : X^{\omega} \to X$ that is a function from $V(X^{\omega})$ onto $V(X)$ which erases the second coordinates. Then the map $\pi : X^{\omega} \to X$ is a *$G$-covering* of $X$, that is a $|G|$-fold regular covering of $X$ with the covering group $G$. Every regular covering of $X$ can be obtained in such a way.

## Short way to construct coverings

Let $X$ be a graph of genus $g$. Choose a spanning tree $T$ in $X$ and $g$ directed edges $e_1, e_2, \ldots, e_g$ from the compliment $X \setminus T$.
An arbitrary *reduced permutation assignment* $\psi : D(X) \to \mathbb{S}_n$ is uniquely determined by the following conditions:

(i) $\psi(e_i) = \xi_i$, where $\xi_i \in \mathbb{S}_n$ for $i = 1, 2, \ldots, g$ and $\psi(e) = 1$, for any edge $e$ which is in $T$;

(ii) $\xi_1, \xi_2, \ldots, \xi_g$ generate a transitive subgroup in $\mathbb{S}_n$.

Then the permutation derived graph gives a required covering.
All connected $n$-fold coverings can be obtained in such a way. Two tuples $(\xi_1, \xi_2, \ldots, \xi_g)$ and $(\xi'_1, \xi'_2, \ldots, \xi'_g)$ give equivalent coverings if and only if there exists $h \in \mathbb{S}_n$ such that $\xi'_i = h\,\xi_i\,h^{-1}$ for all $i = 1, 2, \ldots, g$.

**Monodromy group and covering group**

The tranitive group $\mathrm{Mon}(\psi) = \langle \xi_1, \xi_2, \ldots, \xi_g \rangle$ is called the *monodromy group* of the covering $\psi$. Is has the following properties.

(i) Covering $\psi$ is regular if and only if the group $\mathrm{Mon}(\psi)$ is regular, that is acts without fixed point on the set $\{1, 2, \ldots, n\}$;

(ii) In the case of regular covering $\mathrm{Cov}(\psi) \cong \mathrm{Mon}(\psi)$;

(iii) In the case of irregular covering one can use an isomorphism $\mathrm{Cov}(\psi) \cong C_{\mathbb{S}_n}(\mathrm{Mon}(\psi))$.

**Coverings and transitive homomorphisms**

Let $\Gamma = \pi_1(X, x)$ be the fundamental group of a graph $X$ at vertex $x$. It is well known that there is a one-to-one correspondence between the classes of equivalent $n$-fold coverings of $X$ and the equivalence classes of transitive homomorphisms from $\Gamma$ to the symmetric group $\mathbb{S}_n$ on $n$ symbols. Recall that a homomorphism to $\mathbb{S}_n$ is called *transitive* if its image is a transitive subgroup in $\mathbb{S}_n$. Two homomorphisms, $\theta, \theta' : \Gamma \to \mathbb{S}_n$ are said to be *equivalent* if there exists $h \in \mathbb{S}_n$ such that $\theta' = h\,\theta\,h^{-1}$. [A. Hatcher, Algebraic Topology, Cambridge Univ. Press, Cambridge, 2002, p. 68].

## Coverings and transitive homomorphisms

Let $X$ be a graph of genus $g$. Then $\Gamma$ is a free group of rank $g$. Suppose that $\Gamma$ is freely generated by the elements $x_1, x_2, \ldots, x_g$. Then an arbitrary transitive homomorphism $\theta : \Gamma \to \mathbb{S}_n$ is uniquely determined by the following conditions:

(i) $\theta(x_i) = \xi_i$, where $\xi_i \in \mathbb{S}_n$ for $i = 1, 2, \ldots, g$.

(ii) $\xi_1, \xi_2, \ldots, \xi_g$ generate a transitive subgroup in $\mathbb{S}_n$.

Two homomorphisms defined by tuples $(\xi_1, \xi_2, \ldots, \xi_g)$ and $(\xi_1', \xi_2', \ldots, \xi_g')$ are equivalent if and only if exists $h \in \mathbb{S}_n$ such that $\xi_i' = h\,\xi_i\,h^{-1}$ for all $i = 1, 2, \ldots, g$.

**Coverings and the fundamental group**

If $\varphi : X \to Y$ is a covering and $\varphi(x) = y$ then there is a natural imbedding of the fundamental groups $\varphi_* : \pi_1(X, x) \to \pi_1(Y, y)$ induced by $\varphi$. Moreover, the index of subgroup $\varphi_*\pi_1(X, x)$ in $\pi_1(Y, y)$ coincides with the number of sheets of the covering. The covering $\varphi$ is regular if and only if $\varphi_*\pi_1(X, x)$ is a normal subgroup in $\pi_1(Y, y)$. In the latter case, $Cov(\varphi)$ is canonically isomorphic to the factor-group $\pi_1(Y, y)/\varphi_*\pi_1(X, x)$. The coverings $\varphi : X \to Y$ and $\varphi' : X' \to Y$ are equivalent if and only if the corresponding subgroups $\varphi_*\pi_1(X, x)$ and $\varphi'_*\pi_1(X', x')$ are conjugate in $\pi_1(Y, y)$.

**Coverings and the fundamental group**

Let $Y$ be a graph with fundamental group $\Gamma = \pi_1(Y, y)$ and $H < \Gamma$ be an arbitrary subgroup of $\Gamma$. Then there exists a covering $\varphi : X \to Y$ is a covering with $\varphi(x) = y$ such that $H \cong \pi_1(X, x)$.

# The Schreier formula for Graph Coverings

Let $X$ be a finite connected graph. We define the (homological) genus of $X$ by the formula

$$g(X) = e(X) - v(X) + 1,$$

where $e(X)$ and $v(X)$ are the number of edges and vertices of $X$ respectively. This number is also known as Betti number of cyclotomic index of a graph.

Also, $g(X)$ coincides with the rank of $H_1(X)$ and the rank of $\pi_1(X, x)$.

**Theorem**

*Let $\varphi : X \to Y$ be an n-fold covering of graphs. Then*

$$g(X) - 1 = n(g(Y) - 1).$$

**Proof.** Each edge and each vertex of $Y$ has exactly $n$ preimages in $X$. Hence, $e(X) = ne(Y), v(X) = nv(Y)$ and

$$g(X) - 1 = e(X) - v(X) = ne(Y) - nv(Y) = n(e(Y) - v(Y)) = n(g(Y) - 1).$$

**Universal covering**

The covering $\tilde{Y}$ corresponding to the trivial subgroup $H = \{e\} < \pi_1(Y, y)$ is called the *universal covering.* It has the following properties.

(a) The unversal covering $\tilde{Y}$ exists for any connected graph $Y$ and is uniquely determined up to equivalency.

(b) The universal covering of a graph is a tree.

(c) Let $\varphi : X \to Y$ be a covering. Then there exists a covering $\psi : \tilde{Y} \to X$ such that the composition $\varphi \circ \psi : \tilde{Y} \to Y$ is the universal covering of $Y$.

(d) The group $\Gamma = \pi_1(Y, y)$ acts freely on $\tilde{Y}$ in such a way that the factor graph $\tilde{Y}/\Gamma$ is isomorphic to $Y$.

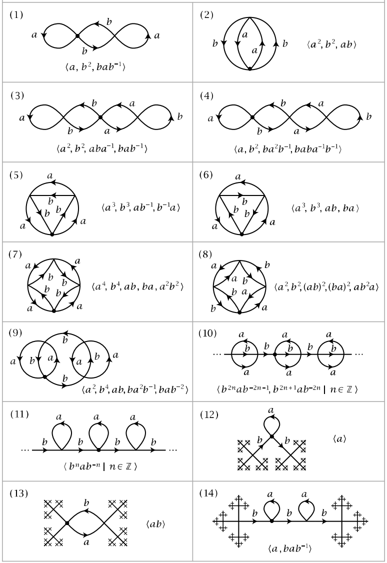
**Coverings and uniformisation theory**

If $\varphi : X \to Y$ is a covering and $\varphi(x) = y$ and is $H = \varphi_\star : \pi_1(X, x) \to \Gamma = \pi_1(Y, y)$ the natural imbedding of the fundamental groups induced by $\varphi$. Denote by $\tilde{Y}$ the universal covering of $Y$. Then there is a free action of groups $\Gamma$ and $H$ on $\tilde{Y}$ such that $\tilde{Y}/\Gamma$ is isomorphic to $Y$, $\tilde{Y}/H$ is isomorphic to $X$ and the covering

$$\varphi : X = \tilde{Y}/H \to Y = \tilde{Y}/\Gamma$$

is induced by the group inclusion $H < \Gamma$.

Some Covering Spaces of $S^1 \vee S^1$



## Exercises

**Exercise 3.1.**
Draw all 2-fold coverings of the figure-eight graph. Show that all of them are regular.

**Exercise 3.2.**
Draw all 3-fold coverings of the figure-eight graph. How many of them are regular?

**Exercise 3.3.**
Let $X$ be a connected graph and $X$ is not a tree. Show that $G$ has infinitely many non-equivalent coverings.

**Exercise 3.4.**
Construct the universal covering tree for the following graphs:

1° Cyclic graph $C_n$,

2° The figure eight graph.

**Exercise 3.5.**
Show that two cyclic graphs $C_m$ and $C_n$ share a finite sheeted covering.

**Exercise 3.6.**
Describe all coverings of a cyclic graph $C_n$.

**Exercise 3.7.**
Let $Y$ be a bipartite graph and $\varphi : X \to Y$ is a graph covering. Show that $X$ is also a bipartite graph.

**Exercise 3.8.**
Let $\varphi : X \to Y$ and $\psi : Y \to Z$ be regular graph coverings. Is it true that $\psi \circ \varphi : X \to Z$ is also regular graph covering?