Branched coverings of graphs

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Definitions and basic properties

In this section we introduce the notion of a graph with multiple edges, loops and semi-edges. That is a slightly more general notion of a graph. This gives us a way to define the action of group on the graph with multiple edges and loops. Also, we are interesting in the group actions with fixed edges, as well as with reversible edges. The factor space of such an action, in general, is not necessary a graph. But, it can be recognised as a graph with semi-edges.

Harmonic Maps

Following (Malnič A., Nedela R., Škoviera M., 2000) we define a *graph with semi-edges* is an ordered quadruple $X = (D, V; I, \lambda)$ where $D = D(X)$ is a set of *darts*, $V = V(X)$ is a nonempty set of *vertices*, which is required to be disjoint from D, I is a mapping of D onto V, called the *incidence function*, and λ is an involutory permutation of D, called the *dart-reversing* involution. For convenience or if λ is not explicitly specified we sometimes write $\bar{x}$ instead of λx. Intuitively, the mapping I assigns to each dart its *initial vertex*, and the permutation λ interchanges a dart and its reverse. The *terminal vertex* of a dart x is the initial vertex of λx. The 2-orbits of λ are called *edges*. The 1-orbits of λ are called *semi-edges*. An edge is called a *loop* if $\lambda x \neq x$ and $I\lambda x = Ix$.

Graphs with semiedges

We identify the set of edges $E(X)$ of X with the following set of unordered pairs of darts:

$$E(X) = \{\{x, \bar{x}\} : x \in D(X),\ x \neq \bar{x}\}.$$

We will refer to the vertices Ix and $I\bar{x}$ as *endpoints* of the edge $\{x, \bar{x}\}$. In a similar way, the set of semi-edges $S(X)$ of X is identified with the set

$$S(X) = \{\{x\} : x \in D(X),\ x = \bar{x}\}.$$

A *directed* edge of X is an ordered pair $(x, \bar{x})$, where $x \in D(X)$ and $x \neq \bar{x}$. We note that all edges $\{x, \bar{x}\} \in E(X)$, including loops, are provided by two directed edges $(x, \bar{x})$ and $(\bar{x}, x)$.

Graphs with semiedges

A *morphism of graphs* $f : X = (D, V; I, \lambda) \to X' = (D', V'; I', \lambda')$ is a function $f : D \cup V \to D' \cup V'$ such that
$f(D) \subseteq D'$, $f(V) \subseteq V'$, $fI = I'f$ and $f\lambda = \lambda' f$. Thus, a morphism is an incidence-preserving mapping which takes vertices to vertices and edges to edges or semi-edges. Note that the image of an edge can be an edge, a loop or a semi-edge, the image of a loop can be a loop or a semi-edge, and the image of a semi-edge can be just a semi-edge.
A bijective morphism $f : X \to X'$ is called an *isomorphism,* and an isomorphism of X onto itself is called an *automorphism.* We will refer to X as a *graph* if it has the empty set of semi-edges $S(X) = \emptyset$.

Graphs with semiedges

The group $\mathrm{Aut}(X)$ of automorphisms of X is a subgroup of $\mathbf{S}_{D(X)}$ leaving invariant each of the sets $V(X)$, $E(X)$, $S(X)$ and preserving incidence. We say that a group G *acts* on X if G is a subgroup of $\mathrm{Aut}(X)$.
Let X be a finite connected graph. We define the *genus* of X to be the number

$$g(X) = 1 - |V(X)| + |E(X)|.$$

Recall that $g(X)$ coincides with the Betti number of X that is the rank of the first homology group $\mathrm{H}_1(X, \mathbb{Z})$. Let G be a finite group acting on the graph X. An edge $\{x, \bar{x}\} \in E(X)$ is said to be *invertible* (or *reversible*) by G if there is an element $g \in G$ such that g sends x to $\bar{x}$ and $\bar{x}$ to x. An edge $\{x, \bar{x}\} \in E(X)$ is said to be *fixed* by G if there is a non-trivial element $g \in G$ that fixes x and $\bar{x}$. We say that G acts on X *without edge reversing* if X has no edges invertible by G. Also, G acts on X *without fixed edges* if X has no edges fixed by G.

Graphs with semiedges

From now on all graphs are supposed to be finite and connected. However, multiple edges, loops and even semiedges are allowed. In this section our target is introduction of the concept a harmonic morphism between graphs generalising the concept of the graph covering studied in previous sections. Equivalently, we are going to introduce the notion of "branched covering between graphs".

Let $\varphi : X \to Y$ be morphism of graphs. Denote by $\mathrm{St}_X(v)$ the *star* formed by vertex v and all darts of X incedent to v. We say that φ is *uniform at a vertex* v of X if there exists a non-negative integer m_v such that for every dart $x \in \mathrm{St}_Y(\varphi(v)) \setminus S(Y)$, we have $|\varphi^{-1}(y) \cap \mathrm{St}_X(v)| = m_v$. A morphism $\varphi : X \to Y$ is said to be *locally uniform* if it is uniform at each vertex of X. For a given locally uniform morphism defined on X, the number m_v attached to the vertex v of X will be called the *multiplicity* at v.

Harmonic morphisms

An uniform morphism $\varphi : X \to Y$ is called to be *harmonic* if it is onto. A group G acts harmonically if G acts fixed point free on the set of directed edges $D(X)$ of a graph X.

Scott Corry and Roman Nedela made the following useful observation

If a group G acts harmonically on a graph X then the canonical projection $X \to X/G$ is a harmonic mophism.

That gives us a lot of non-trivial examples of harmonic morphisms.

Harmonic morphisms

The following lemma plays a crucial role in the theory of uniform morphisms.

Lemma

Let $\varphi : X \to Y$ be a locally uniform morphism. Then there exists a constant $d = deg(\varphi)$ (depending just on φ) such that the size of the fibre $|\varphi^{-1}(x)| = d$ for every ordinary dart $x \in D(Y) \setminus S(X)$. Furthermore, $d = \sum_{v \in \varphi^{-1}(w)} m_v$ for any vertex v of Y.

Proof. Let $Y = (D, V; I, \lambda)$. Suppose to the contrary that there are two ordinary darts x and y in Y such that $|\varphi^{-1}(x)| = |\varphi^{-1}(y)|$.
Since φ is locally uniform, the darts x and y must be based at different vertices $u \neq v$. Since Y is connected, we may choose x and y such that u and v are adjacent, and let $e = \{z, \bar{z}\}$ be an edge joining u to v.
As far as φ is locally uniform $|\varphi^{-1}(x)| = |\varphi^{-1}(z)|$ and $|\varphi^{-1}(y)| = |\varphi^{-1}(\bar{z})|$. Since φ is a morphism, we have $|\varphi^{-1}(z)| = |\varphi^{-1}(\bar{z})|$ and therefore, $|\varphi^{-1}(x)| = |\varphi^{-1}(y)|$. Contradiction.

The number $deg(\varphi)$ from the above Lemma will be called the *degree* of a locally uniform morphism φ. It follows that the degree determines the size of the fibre $\varphi^{-1}(e)$ over each edge $e = \{x, \bar{x}\} \in E(Y)$.

Harmonic morphisms

A locally uniform morphism of degree 0 will be called *trivial.* Trivial locally uniform morphisms $\varphi : X \to Y$ are characterised by the following properties: Y has just one vertex, and every dart of X is sent into a semi-edge. Note that trivial harmonic morphisms are exactly the graph epimorphisms onto stars.

Riemann-Hurwitz formula for harmonic morphisms

Let $e(X)$ denote the number of edges in X (the semi-edges are not counted in $e(X)$), and let $v(X)$ denote the number of vertices of X. By the *genus* of a graph X we mean the integer $g(X) = e(X) - v(X) + 1$. If a branched covering takes a closed surface S onto a closed surface S', then the well known Riemann-Hurwitz formula relates the genera of the two surfaces, degree and branch indices of the covering.

A similar statement holds for graphs and harmonic morphisms. Before proceeding further, we need the following definition.

Define the multiplicity of edges of X in a harmonic morphism $X \to Y$ by setting $m_e = 2$, if e is mapped onto a semi-edge, and setting $m_e = 1$ otherwise.

Theorem

(Riemann-Hurwitz formula) Let $\varphi : X \to Y$ be a harmonic morphism, and let $g = g(X)$ and $\gamma = g(Y)$ are the respective genera of X and Y. Then

$$g - 1 = \deg(\varphi)(\gamma - 1) + \sum_{v \in V}(m_v - 1) + \sum_{e \in E}(m_e - 1),$$

where V, E is the set of vertices and the set of edges of X respectively.

Proof. Since φ is harmonic, we have $\sum\limits_{v \in V} m_v = \deg(\varphi)\, v(Y)$, where $v(Y)$ is the number of vertices of Y. Inserting $\gamma - 1 = e(Y) - v(Y)$, where $e(Y)$ is the number of edges in Y, we rewrite the right side of the equation as

$$\deg(\varphi)(\gamma - 1) + \sum_{v \in V}(m_v - 1) + \sum_{e \in E}(m_e - 1) =$$

$$\deg(\varphi)(e(Y) - v(Y)) + \deg(\varphi)v(Y) - v(X) + \sum_{e \in E}(m_e - 1) =$$

$$\deg(\varphi)e(Y) + \sum_{e \in E}(m_e - 1) - v(X).$$

Riemann-Hurwitz formula for harmonic morphisms

To complete the proof we need to prove that

$$\deg(\varphi)\, e(Y) + \sum_{e \in E} (m_e - 1) = e(X). \tag{1}$$

Each edge in Y lifts to $deg(\varphi)$ edges in X. Since the sum $\sum\limits_{e \in E} (m_e - 1)$ counts the number of edges in X mapped onto semi-edges and the map φ is onto, we get the required equality (1).

Riemann-Hurwitz formula for graph coverings

The following statement immediately follows from the Riemann-Hurwitz formula.

Theorem (Schreier formula)

Let $\varphi : G \to G'$ be a graph covering. Suppose that G and G' are graphs of genera g and g' respectively. Then we have

$$g - 1 = deg(\varphi)(g' - 1).$$

Riemann-Hurwitz formula for harmonic morphisms

Let graph X has no loops and no semi-edges and Y has no loops, but, possibly, has semi-edges. Consider a harmonic morphism $\varphi : X \to Y$. An edge e of X is called *vertical* if its both ends are mapped in the same vertex of Y. Then the sum $N = \sum_{e \in E}(m_e - 1)$ is equal to the number of vertical edges of Y. As a consequence of the above theorem we have the following result obtained earlier by M. Baker and S. Norine (2009).

Corollary

Let $\varphi : X \to Y$ be a harmonic morphism, and let $g = g(X)$ and $\gamma = g(Y)$ are the respective genera of X and Y. Then

$$g - 1 = \deg(\varphi)(\gamma - 1) + \sum_{v \in V}(m_v - 1) + N,$$

where V and E, respectively, are the set of vertices and the set of edges of X, and N is the number of vertical edges of X.

Groups acting on a graph without edge reversing

Our next result is the following theorem for groups acting on a graph, possibly with fixed edges, but without edge reversing.

Theorem

Let X be a graph of genus g and G is a finite group acting on X without edge reversing. Denote by $g(X/G)$ genus of the factor graph X/G. Then

$$g-1=|G|(g(X/G)-1)+\sum_{v\in V(X)}(|G^v|-1)-\sum_{e\in E(X)}(|G^e|-1),$$

where $V(X)$ is the set of vertices, $E(X)$ is the set of edges of X, G^x stands for the stabiliser of $x\in V(X)\cup E(X)$ in G and $|G^x|$ is the order of a stabiliser.

Proof. Since G acts on X without edge inversions, the factor graph X/G is well defined. The vertices of X/G are orbits Gv, $v \in V(X)$, while the edges are orbits Ge, $e \in E(X)$. Vertices Gv_1 and Gv_2 are incident to an edge Ge in X/G if and only if v_1 and v_2 are incident to the edge e in X. Prescribe to every $\tilde{x} \in V(X/G) \cup E(X/G)$ a group $G_{\tilde{x}}$ isomorphic to G^x, where x is one of the preimages $\tilde{x}$ under the canonical map $\varphi : X \to X/G$. Since G acts transitively of fibres of φ the group $G_{\tilde{x}}$ is well defined. One can consider the graph X/G with prescribed groups G_v, $v \in V(X/G)$ and G_e, $e \in V(X/G)$ as a *graph of groups* in sense of the Bass-Serre theory. We note that the fibre $\varphi^{-1}(\tilde{x})$ of $\tilde{x}$ consists of $\frac{|G|}{|G_{\tilde{x}}|}$ elements. Hence,

$$|V(X)| = \sum_{v \in V(X)} 1 = \sum_{\tilde{v} \in V(X/G)} \frac{|G|}{|G_{\tilde{v}}|} \tag{2}$$

and

$$|E(X)| = \sum_{e \in E(X)} 1 = \sum_{\tilde{e} \in V(X/G)} \frac{|G|}{|G_{\tilde{e}}|}. \tag{3}$$

By definition of genus from (2) and (3) we obtain

$$
\begin{aligned}
g-1 &= |E(X)|-|V(X)| = \sum_{\bar{e}\in V(X/G)} \frac{|G|}{|G_{\tilde{e}}|} - \sum_{\tilde{v}\in V(X/G)} \frac{|G|}{|G_{\tilde{v}}|} \\
&= |G|\Big(\sum_{\tilde{e}\in E(X/G)} 1 - \sum_{\tilde{v}\in V(X/G)} 1\Big) \\
&+ \sum_{\tilde{e}\in E(X/G)} \frac{|G|}{|G_{\tilde{e}}|}(1-|G_{\bar{e}}|) - \sum_{\tilde{v}\in V(X/G)} \frac{|G|}{|G_{\tilde{v}}|}(1-|G_{\tilde{v}}|) \\
&= |G|(g(X/G)-1) + \sum_{e\in E(X)} (1-|G^e|) - \sum_{v\in V(X)} (1-|G^v|) \\
&= |G|(g(X/G)-1) + \sum_{v\in V(X)} (|G^v|-1) - \sum_{e\in E(X)} (|G^e|-1).
\end{aligned}
$$

Groups acting on a graph with edge reversing

In this section we prove the following theorem.

Theorem

Let X be a graph of genus g and G is a finite group acting on X, possibly with edge reversing. Denote by $g(X/G)$ genus of the factor graph (X/G). Then

$$g-1 = |G|(g(X/G)-1) + \sum_{v\in V(X)} (|G^v|-1) - \sum_{e\in E(X)} (|G^e|-1) + \sum_{e\in E^{inv}(X)} |G^e|,$$

where $V(X)$ is the set of vertices, $E(X)$ is the set of edges of X, G^x is the stabiliser of $x \in V(X) \cup E(X)$ in G, and $E^{inv}(X)$ is the set of invertible edges of X.

Proof. Consider G as a directed graph prescribing to each edge $e \in E(X)$ the endpoints $\partial_0 e$ and $\partial_1 e$. Let $\bar{e}$ be the edge e with the opposite orientation, that is $\partial_i \bar{e} = \partial_{1-i} e$, $i = 1, 2$. Denote by $\partial_{1/2} e$ the white vertex of the barycentric subdivision X' that divides an edge e in two edges $h_e = (\partial_0 e, \partial_{1/2} e)$ and $h_{\bar{e}} = (\partial_{1/2} e, \partial_1 e)$.
Since G acts on X' without edge reversing, by the previous theorem we have

$$g - 1 = |G|(g(X'/G) - 1) + \sum_{v \in V(X')} (|G^v| - 1) - \sum_{e \in E(X')} (|G^e| - 1). \quad (4)$$

Groups acting on a graph with edge reversing

The set of vertices $V(X')$ of the bipartite graph X' the union $V(X') = B(X') \cup W(X')$ of the sets of black and white vertices, where

$$B(X') = V(X) \text{ and } W(X') = \{\partial_{1/2}e : \ e \in E(X)\}.$$

The stabiliser of a point $v = \partial_{1/2}e$ in G consists of the elements of G that permute the endpoints of e leaving e invariant or the ones that fix e. Hence, $G^{\partial_{1/2}e} = G^{\{e\}}$, where $G^{\{e\}}$ is the setwise stabiliser of the set $e = \{x, \bar{x}\}$ in G. As a result, we obtain

$$\begin{aligned} \sum_{v\in V(X')} (|G^v| - 1) &= \sum_{v\in V(X)} (|G^v| - 1) + \sum_{e\in E(X)} (|G^{\partial_{1/2}e}| - 1) \\ &= \sum_{v\in V(X)} (|G^v| - 1) + \sum_{e\in E(X)} (|G^{\{e\}}| - 1). \end{aligned} \quad (5)$$

Groups acting on a graph with edge reversing

For each $e \in E(X')$ we have $G^e = G^{h_e} = G^{h_{\bar{e}}}$. Hence

$$\begin{aligned} \sum_{e\in E(X')} (|G^e| - 1) &= \sum_{e\in E(X)} (|G^{h_e}| - 1) + \sum_{e\in E(X)} (|G^{h_{\bar{e}}}| - 1) \\ &= 2 \sum_{e\in E(X)} (|G^e| - 1). \end{aligned} \tag{6}$$

Substituting equations (5) and (6) into (4) we obtain

$$\begin{aligned} g - 1 &= |G|(g(X'/G) - 1) + \sum_{v\in V(X)} (|G^v| - 1) - \sum_{e\in E(X)} (|G^e| - 1) \\ &+ \sum_{e\in E(X)} (|G^{\{e\}}| - |G^e|). \end{aligned} \tag{7}$$

Groups acting on a graph with edge reversing

Denote by $E^{inv}(X)$ the set of invertible edges of X. Then $|G^{\{e\}}| = 2|G^e|$ if $e \in E^{inv}(X)$ and $|G^{\{e\}}| = |G^e|$ otherwise. The smoothing of a white vertex in graph X'/G decreases the number of vertices and the number of edges of the graph by one. So, it does not affect the genus $g(X'/G) = 1 - |V(X'/G)| + |E(X'/G)|$. Hence, by definition of $(X/G)_{tail}$ we have $g(X/G)_{tail} = g(X'/G)$. Then (7) can be rewritten in the form

$$\begin{aligned} g - 1 &= |G|(g(X/G)_{tail} - 1) + \sum_{v \in V(X)} (|G^v| - 1) - \sum_{e \in E(X)} (|G^e| - 1) \\ &+ \sum_{e \in E^{inv}(X)} |G^e|. \end{aligned} \tag{8}$$

Harmonic Maps and Graphs of Groups

From now on we restrict ourself by harmonic maps of graphs without semi-edges. Them we employ the Bass-Serre theory of graphs of groups to prove uniformisation theorems for this class of maps.

Following H. Bass we define a *graph of groups* to be a pairs $\mathbb{A} = (A, \mathcal{A})$, where A is a connected graph, and $\mathcal{A} = \{A_a\}_{a \in A}$ assigns group A_a to each vertex $a \in A$.

Let $\mathbb{A} = (A, \mathcal{A})$ and $\mathbb{A}' = (A', \mathcal{A}')$ be graphs of groups. By a *covering of graph of groups*

$$\mathbb{F} = (\varphi, \Phi) : \mathbb{A} \to \mathbb{A}'$$

we mean

(i) a harmonic morphism $\varphi : A \to A'$;

(ii) a set Φ of injective homomorphisms

$$\varphi_a : \mathcal{A}_a \to \mathcal{A}'_{\varphi(a)} \ (a \in A) \text{ such that } m_\varphi(a)|\mathcal{A}_a| = |\mathcal{A}'_{\varphi(a)}|,$$

where $m_\varphi(a)$ is the multiplicity of φ at the point a.

The *fundamental group* of a graph of group $\mathbb{A} = (A, \mathcal{A})$, denoted $\pi_1(\mathcal{A})$, is defined as the free product

$$(*_{a \in A} \mathcal{A}_a) * \pi_1(A),$$

where $\pi_1(A) = \pi_1(A, a)$ denotes the fundamental group of the graph A.

To every graph of groups $\mathbb{A}$ one can associate a *Bass-Serre universal covering tree* $\widetilde{\mathbb{A}}$, which is a tree with $\pi_1(\widetilde{\mathbb{A}}) = \langle 1 \rangle$ that comes equipped with a natural group action of the fundamental group $\pi_1(\mathbb{A})$ without edge-inversions. Moreover, the quotient graph $\widetilde{\mathbb{A}}/\pi_1(\mathbb{A})$ is isomorphic to $\mathbb{A}$.

Bass-Serre uniformization theorem

Theorem (H. Bass, J.-P. Serre)

Let $\mathbb{F} : \mathbb{X} \to \mathbb{Y}$ *be a graph of group covering. Then* $\mathbb{X}$ *and* $\mathbb{Y}$ *share the same universal covering tree* $\widetilde{\mathbb{Y}}$. *Moreover, the groups* $H = \pi_1(\mathbb{X})$ *and* $\Gamma = \pi_1(\mathbb{Y})$ *are acting on* $\widetilde{\mathbb{Y}}$ *in such a way that* $\mathbb{X} \cong \widetilde{\mathbb{Y}}/H$, $\mathbb{Y} \cong \widetilde{\mathbb{Y}}/\Gamma$ *and the covering*

$$\mathbb{F} : \mathbb{X} = \widetilde{\mathbb{Y}}/H \to \mathbb{Y} = \widetilde{\mathbb{Y}}/\Gamma$$

is induced by the group inclusion $H < \Gamma$.