Spanning trees

Alexander Mednykh

Sobolev Institute of Mathematics
Novosibirsk State University

PhD Summer School in Discrete Mathematics
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Spanning tree

A *spanning tree* T of a connected, undirected graph G is a tree composed of all the vertices and some (or perhaps all) of the edges of G. In other words, a spanning tree of G is a selection of edges of G that form a tree spanning every vertex. That is, every vertex lies in the tree, but no cycles (or loops) are allowed. On the other hand, every bridge of G must belong to T. A spanning tree of a connected graph G can also be defined as a maximal set of edges of G that contains no cycle, or as a minimal set of edges that connect all vertices.

Spanning Trees.

Counting spanning trees

The number $t(G)$ of spanning trees of a connected graph is a well-studied invariant. In some cases, it is easy to calculate $t(G)$ directly. For example, if G is itself a tree, then $t(G) = 1$, while if G is the cycle graph C_n with n vertices, then $t(G) = n$. For any graph G, the number $t(G)$ can be calculated using Kirchhoff's matrix-tree theorem.

Here are some known results concerning counting spanning trees of graphs.

1. Complete graph K_n : $t(K_n) = n^{n-2}$ (Cayley's formula),
2. Complete bipartite graph $K_{n,m}$: $t(K_{n,m}) = m^{n-1}n^{m-1}$,
3. n-dimensional cube graph Q_n : $t(Q_n) = 2^{2^n-n-1}\prod_{k=2}^{n} k^{\binom{n}{k}}$.

Kirchhoff Matrix-Tree Theorem

The celebrated Kirchhoff Matrix-Tree Theorem is the following statement.

Theorem (Kirchhoff (1847))

All cofactors of Laplacian matrix $L(G)$ are equal to $t(G)$.

More convenient form of this result were obtained by A. K. Kel'mans and V. M. Chelnokov.

Theorem (Kel'mans, Chelnokov (1974))

Let $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ denote the eigenvalues of the Laplace matrix $L(G)$ of a n point graph G. Then

$$t(G) = \frac{1}{n} \prod_{k=2}^{n} \lambda_k.$$

The Temperley's formula

One more convenient way to count spanning treses.

Theorem (Temperley, H. N. V. (1964))

The number of spanning trees of a n point graph G is given by the formula

$$t(G) = \det(L(G) + \frac{1}{n^2}J),$$

where J is $n \times n$ matrix all of whose elements are unity.

Now we will present an uniform proof for all the three previous theorems.

Proof of Kirchhoff, Kel'mans - Chelnokov and Temperley theorems

Proof.

Let $L = L(G)$ be the Laplacian matrix of G with eigenvalues $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$. The (i,j)-cofactor of a matrix M is by definition $(-1)^{i+j} \det M(i,j)$, where $M(i,j)$ is the matrix obtained from M by deleting row i and column j.
Let l_{xy} be the (x,y)-cofactor of L. Note that l_{xy} does not depend on an ordering of the vertices of G.
We set $N = t(G)$ and show that

$$N = l_{xy} = \det(L + \frac{1}{n^2}J) = \frac{1}{n}\lambda_2 \ldots \lambda_n \text{ for any } x, y \in V(G).$$

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Let L^S, for $S \subset V(G)$, denote the matrix obtained from L by deleting the rows and columns indexed by S, so that $l_{xx} = \det L^{\{x\}}$. The equality $N = l_{xx}$ follows by induction on n, and for fixed $n > 1$ on the number of edges incident with x. Indeed, if $n = 1$ then $l_{xx} = 1$. Otherwise, if x has degree 0, then $l_{xx} = 0$ since $L^{\{x\}}$ has zero row sums. Now, if xy is an edge, then deleting this edge from G decreases l_{xx} by $\det L^{\{x,y\}}$, which by induction is the number of spanning trees of G with edge xy collapsing to a point, which is the number of spanning trees containing the edge xy. This shows $N = l_{xx}$. Since the sum of the columns of L is zero, so that one column is minus the sum of the other columns, we have $l_{xx} = l_{xy}$ for any x, y.

Now we consider the Laplacian polynomial
$\mu(G,t) = \det(tI - L) = t\prod_{i=2}^{n}(t-\lambda_i)$ for graph G. Then
$(-1)^{n-1}\lambda_2\ldots\lambda_n$ is the coefficient of t, that is, $\frac{d}{dt}\det(tI-L)|_{t=0}$.
We note that

$$\frac{d}{dt}\det(tI-L) = \sum_x \det(tI - L^{\{x\}}).$$

Putting $t=0$ we obtain $\lambda_2\ldots\lambda_n = \sum_x l_{xx} = nN$.

Finally, the eigenvalues of $L+\frac{1}{n^2}J$ are $\frac{1}{n}$ and $\lambda_2,\ldots,\lambda_n$, so

$$\det(L+\frac{1}{n^2}J) = \frac{1}{n}\lambda_2\ldots\lambda_n.$$

Spanning Trees

A generalization of the Matrix-Tree-Theorem was obtained by Kelmans (1967) who gave a combinatorial interpretation to all the coefficients of $\mu(G,x)$ in terms of the numbers of certain subforests of the graph. This result has been obtained even in greater generality (for weighted graphs) by Fiedler and Sedláček.

Theorem (Kel'mans (1967))

If

$$\mu(G,x) = x^n - c_1x^{n-1} + \ldots + (-1)^i c_i x^{n-i} + \ldots + (-1)^{n-1} c_{n-1} x$$

then

$$c_i = \sum_{S\subset V,\,|S|=n-i} t(G_S),$$

where $t(H)$ is the number of spanning trees of H, and G_S is obtained from G by identifying all vertices of S to a single one.

Spanning Trees

From the last theorem we can derive useful corollary.

Corollary

The degree of Laplacian polynomial $\mu(G, x)$ is equal to $n = |V(G)|$. Its coefficients c_1 and c_{n-1} are given by the formulas $c_1 = 2|E(G)|$ and $c_{n-1} = |V(G)| \cdot t(G)$.

Hence, the number of vertices $|V(G)|$, number of edges $|E(G)|$ and the number of spanning trees $t(G)$ are uniquely defined by the Laplacian polynomial.

Some recursive formulas for t(G)

Now we give a few recursive formulas for the number of spanning trees employed in graph theory by W. Feussner (1904) and J. W. Moon (1970).

Denote by $G - e$ the graph obtained by removing edge e from the graph G. Let $G\backslash e$ be the graph obtained from graph G by contracting edge e. In other words, $G\backslash e$ is obtained by deleting edge e and identifying its ends. Then the following formula takes a place.

$$t(G) = t(G - e) + t(G\backslash e).$$

Proof. We note that the set of spanning trees of a given graph G decomposed in two disjoint sets. First set consist of tree containing selected edge $e \in E(G)$ and second set consist of trees that do not contain e. The number of spanning trees that contains e is exactly $t(G\backslash e)$ because each of them corresponds to a spanning tree of $G\backslash e$. The number of spanning tree that do not contain e is $t(G - e)$, since each of them is also a spanning tree of $G - e$ and vice versa.

Denote by $G_{s,e}$ the graph resulting from subdivision of an edge e of a graph G. Then

$$t(G_{s,e}) = t(G \backslash e) + 2t(G - e) = t(G) + t(G - e).$$

Proof. Again, the number of spanning trees of X that contains e is $t(G \backslash e)$. All of them also the spanning trees of $t(G_{s,e})$. If a spanning tree of G do not contains e then it can be extended to be a spanning tree of $t(G_{s,e})$ in two different ways. Hence, $t(G_{s,e}) = t(G \backslash e) + 2t(G - e)$. By the previous statement $t(G \backslash e) + 2t(G - e) = t(G) + t(G - e)$.

Let $G_{p,e}$ denotes the results of adding an edge in parallel an edge e of a graph G. Then

$$t(G_{p,e}) = t(G) + t(G \backslash e).$$

Proof. Distinguishing spanning trees that contain an edge e and that are not we have

$$t(G_{p,e}) = t(G-e) + 2t(G \backslash e) = t(G) + t(G \backslash e).$$

Let G_1 and G_2 are the graphs with exactly one vertex in common. Then

$$t(G_1 \cup G_2) = t(G_1) \cdot t(G_2),$$

where $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$.

Proof. Let T be a spanning tree of $G_1 \cup G_2$. Then $T_1 = T \cap G_1$ and $T_2 = T \cap G_2$ are spanning trees for G_1 and G_2 respectively. Moreover $G_1 \cap G_2 = \{v\}$, where v is the common vertex of G_1 and G_2. Conversely, let T_1 and T_2 are respective spanning trees for G_1 and G_2. Then $T_1 \cap T_2 = \{v\}$ and $T = T_1 \cup T_2$ is a spanning tree of $G_1 \cup G_2$.

Chebyshev polynomials

The Chebyshev polynimial of the first kind is defined by the formula

$$T_n(x) = \cos(n \arccos x).$$

Eqivalently,

$$T_n(x) = \frac{(x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n}{2}.$$

Also, $T_n(x)$ satisfies the recursive relation

$$T_0(x) = 1,\ T_1(x) = x,\ T_n(x) = 2x \cdot T_{n-1}(x) - T_{n-2}(x),\ n \geq 2.$$

Chebyshev polynomials

The Chebyshev polynimial of the second kind is defined by the formula

$$U_n(x) = \frac{\sin((n+1)\arccos x)}{\sin(\arccos x)}.$$

Equivalently,

$$U_n(x) = \frac{(x+\sqrt{x^2-1})^{n+1} - (x-\sqrt{x^2-1})^{n+1}}{2\sqrt{x^2-1}}.$$

Also, $U_n(x)$ satisfies the recursive relation

$$U_0(x) = 1,\ U_1(x) = 2x,\ U_n(x) = 2x \cdot U_{n-1}(x) - U_{n-2}(x),\ n \geq 2.$$

We have $U_n(\cos\frac{k\pi}{n+1}) = 0, k = 1, 2, \ldots, n$. Hence

$$U_n(x) = 2^n \prod_{k=1}^{n} (x - \cos\frac{k\pi}{n+1}).$$

Since

$$U_n(x) = (-1)^n U_n(-x) = 2^n \prod_{k=1}^{n} (x + \cos \frac{k\pi}{n+1})$$

we obtain

$$U_n^2(x) = \prod_{k=1}^{n} (4x^2 - 4\cos^2 \frac{k\pi}{n+1}).$$

Polynomials $T_n(x)$ and $U_{n-1}(x)$ are related by the following identity

$$T_n^2(x) + (x^2 - 1)U_{n-1}^2(x) = 1.$$

Chebyshev polynomials

Consider an $n \times n$ matrix

$$A_n(x) = \begin{pmatrix} x & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 2x & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 2x & 1 & \dots & 0 & 0 \\ 0 & 0 & 1 & 2x & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 2x & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & 2x \end{pmatrix}.$$

Then $\det A_n(x) = T_n(x)$, (Nash, 1986).

Chebyshev polynomials

In a similar way, if

$$B_n(x) = \begin{pmatrix} 2x & -1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 2x & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2x & -1 & \dots & 0 & 0 \\ 0 & 0 & -1 & 2x & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 2x & -1 \\ 0 & 0 & 0 & 0 & \dots & -1 & 2x \end{pmatrix},$$

then $\det B_n(x) = U_n(x)$.

Exercises

Exercise 2.1.
Prove that the number of spanning trees for the path graph P_n is 1.

Exercise 2.2.
Prove that the number of spanning trees for the cyclic graph C_n is n.

Exercise 2.3.
Prove the Cayley formula for the number of spanning trees for the complete graph K_n : $t(K_n) = n^{n-2}$.

Exercise 2.4.
Prove that the number of spanning trees for the complete bipartite graph $K_{n,m}$ is given by the formula $t(K_{n,m}) = m^{n-1} n^{m-1}$.

Exercise 2.5. Denote by $G - e$ the graph obtained by removing edge e from the graph G. Let $G \backslash e$ be the graph obtained from graph G by contracting edge e. In other words, $G \backslash e$ is obtained by deleting edge e and identifying its ends. Prove the following formula

$$t(G) = t(G - e) + t(G \backslash e).$$

Exercise 2.6. Denote by $G_{s,e}$ the graph resulting from subdivision of an edge e of a graph G. Then

$$t(G_{s,e}) = t(G \backslash e) + 2t(G - e) = t(G) + t(G - e).$$

Exercises

Exercise 2.7.

Find the number of spanning trees for the wheel graph $W_n = K_1 * C_n$.
Answer: If n is odd then $t(W_n) = \ell_k^2$, if n is even then $t(W_n) = 5f_k^2$, where ℓ_j is j-th Lukas number and f_k is k-th Fibonacci number.
Note:
$\ell_1 = 1,\ \ell_2 = 3,\ \ell_{k+2} = \ell_{k+1} + \ell_k,\ k \geq 1.$
$f_1 = 1,\ f_2 = 1,\ f_{k+2} = f_{k+1} + f_k,\ k \geq 1.$
$f_{2n} = \ell_n \cdot f_n$ and $\ell_n = f_{n-1} + f_{n+1}.$

Exercise 2.8.
Find the number of spanning trees for the fan graph $F_n = K_1 * P_n$.

Solution: By Exercise 1.7 and Kel'mans-Chelnokov theorem we obtain

$$t(F_n) = \prod_{k=1}^{n-1}(3 - 2\cos\frac{\pi k}{n}) = 2^{n-1}\prod_{k=1}^{n-1}(\frac{3}{2} - \cos\frac{\pi k}{n}).$$

$$= U_{n-1}(\frac{3}{2}) = \frac{1}{\sqrt{5}}\Big((\frac{3+\sqrt{5}}{2})^n - (\frac{3-\sqrt{5}}{2})^n\Big)$$

$$= \frac{1}{\sqrt{5}}\Big((\frac{1+\sqrt{5}}{2})^{2n} - (\frac{1-\sqrt{5}}{2})^{2n}\Big) = f_{2n}.$$

Exercise 2.9.
Find the number of spanning trees for the lattice graph $L_{m,n} = K_m \times K_n$.

Solution: The Laplace spectrums of the graphs K_m and K_n are $\mu_0 = 0,\ \mu_i = m,\ i = 1, \ldots, m-1$ and $\lambda_0 = 0,\ \lambda_j = n,\ j = 1, \ldots, n-1$.

Then $t(K_m \times K_n) = \frac{1}{m\,n} \prod\limits_{\substack{i=0 \\ i+j>0}}^{m-1} \prod\limits_{j=0}^{n-1} (\mu_i + \lambda_j) =$

$$\frac{1}{m\,n} \prod_{j=1}^{n-1} \lambda_j \prod_{i=1}^{m-1} \mu_i \prod_{i=1}^{m-1} \prod_{j=1}^{n-1} (\lambda_j + \mu_i) = m^{m-2} n^{n-2} (m+n)^{(m-1)(n-1)}.$$

Exercise 2.10.
Prove that the following result by Boesch and Prodinger

$$t(K_m \times C_n) = \frac{n}{m} 2^{m-1} (T_n(1 + \frac{m}{2}) - 1),$$

where $T_n(x) = \cos(n \arccos x)$ is the Chebyshev polynomial of the first kind.

Solution:

$$t(K_m \times C_n) = \frac{1}{mn} \prod_{\substack{i=0 \\ i+j>0}}^{m-1} \prod_{j=0}^{n-1} (\mu_i + \lambda_j) = \frac{1}{mn} \prod_{j=1}^{n-1} \lambda_j \prod_{i=1}^{m-1} \mu_i \prod_{i=1}^{m-1} \prod_{j=1}^{n-1} (\lambda_j + \mu_i)$$

$$= t(C_n) t(K_m) \prod_{i=1}^{m-1} \prod_{j=1}^{n-1} (m + 2 - 2\cos(2\pi j/n))$$

$$= n\, m^{m-2} (\prod_{j=1}^{n-1} (m + 4 - 4\cos^2(\pi j/n)))^{m-1} = n\, m^{m-2} \left[U_{n-1}^2 (\sqrt{\tfrac{m+4}{4}}) \right]^{m-1}.$$

Exercises

From elementary identities $\sin^2(u) = \frac{1-\cos(2u)}{2}$, $\cos(2u) = 2\cos^2(u) - 1$ and basic definitions of the Chebyshev polynomials one can derive the following relations

$$U_{n-1}^2(x) = \frac{1}{2(1-x^2)}(1 - T_{2n}(x))$$

$$= \frac{1}{2(1-x^2)}(1 - T_n(T_2(x))) = \frac{1}{2(1-x^2)}(1 - T_n(2x^2 - 1)).$$

Putting $x = \sqrt{\frac{m+4}{4}}$ we get the formula by Boesch and Prodinger

$$t(K_m \times C_n) = \frac{n}{m} 2^{m-1} (T_n(1 + \frac{m}{2}) - 1)^{m-1}.$$

Exercise 2.11.
Prove that the number of spanning trees for the prism $P_2 \times C_n$ is given by the formula

$$t(P_2 \times C_n) = \frac{n}{2}((2+\sqrt{3})^n + (2-\sqrt{3})^n - 2).$$

Solution: Since $P_2 = K_2$, we put $m = 2$ in the solution of Exercise 2.10 to obtain

$$t(P_2 \times C_n) = n(T_n(2) - 1) = \frac{n}{2}((2+\sqrt{3})^n + (2-\sqrt{3})^n - 2).$$

Exercise 2.12.
Prove that the number of spanning trees for the Moebius ladder graph M_n is given by the formula

$$t(M_n) = \frac{n}{2}((2+\sqrt{3})^n + (2-\sqrt{3})^n + 2).$$

Solution: Let us note that the Laplacian matrix for M_n is circulant $circ\{v_0 \ldots, v_{2n-1}\}$, where $v_0 = 3,\ v_1 = -1,\ v_2 = \ldots = v_{n-1} = 0,$ $v_n = -1,\ v_{n+1} = \ldots = v_{2n-2} = 0,\ v_{2n-1} = -1$. Let $\varepsilon = e^{\frac{2\pi i}{2n}}$ be the $2n$-th primitive root of unity.
Then $L(M_n)$ has the following spectrum

$$\lambda_k = \sum_{j=0}^{2n-1} \varepsilon^{kj} v_j = 3 + (-1)^{k+1} - 2\cos\frac{\pi k}{n},\ k = 0, \ldots, 2n-1.$$

Exercises

We have

$$t(M_n) = \frac{1}{2n} \prod_{k=1}^{2n-1} (3 + (-1)^{k+1} - 2\cos\frac{\pi k}{n})$$

$$= \frac{1}{2n} \prod_{j=1}^{n-1} (4 - 2\cos\frac{(2j-1)\pi}{n}) \prod_{j=1}^{n-1} (2 - 2\cos\frac{2j\pi}{n}).$$

We note that

$$\prod_{j=1}^{n-1} (2 - 2\cos\frac{2j\pi}{n}) = \prod_{j=1}^{n-1} (4 - 4\cos^2\frac{j\pi}{n}) = U_{n-1}^2(1) = n^2.$$

Remark:

$$U_{n-1}(1) = \frac{\sin(n \arccos 1)}{\sin(\arccos 1)} = \lim_{u \to 0} \frac{\sin(n\, u)}{\sin(u)} = n.$$

Exercises

Now we simplify the first product

$$\prod_{j=1}^{n-1}(4-2\cos\frac{(2j-1)\pi}{n}) = \prod_{j=1}^{2n-1}(4-2\cos\frac{2j\pi}{2n})\Big/\prod_{j=1}^{n-1}(4-2\cos\frac{2j\pi}{n}).$$

We get

$$\prod_{j=1}^{n-1}(4-2\cos\frac{2j\pi}{n}) = \prod_{j=1}^{n-1}(6-4\cos^2\frac{j\pi}{n}) = U_{n-1}^2(\sqrt{\frac{3}{2}}).$$

From the properties of Chebyshev polynomias we have

$$U_{n-1}^2(x) = \frac{1}{2(1-x^2)}(1-T_{2n}(x)) = \frac{1}{2(1-x^2)}(1-T_n(2x^2-1)).$$

In particular, for $x=\sqrt{3/2}$ we obtain $U_{n-1}^2(\sqrt{\frac{3}{2}}) = T_n(2)-1$. Simillary,

$$U_{2n-1}^2(\sqrt{\frac{3}{2}}) = T_{2n}(2)-1.$$

Exercises

By making use of the identity $\frac{\sin 2n\,u}{\sin n\,u} = 2\cos n\,u$, we have

$$t(M_n) = \frac{1}{2n} \cdot \frac{T_{2n}(2) - 1}{T_n(2) - 1} \cdot n^2 = n(T_n(2) + 1).$$

Equivalently,

$$t(M_n) = \frac{n}{2}((2 + \sqrt{3})^n + (2 - \sqrt{3})^n + 2).$$